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# One Effective Method for Solving Singularly Perturbed Equations

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**Abstract:** Numerical methods are widely used to study the solution of singularly perturbed equations. At the same time, their application to the solution of such equations encounters serious difficulties; they are associated with the presence of a small parameter at the highest derivative and the appearance in the solution area of areas with high frequency-amplitude sawtooth jumps. In this case, the requirements for the efficiency and accuracy of numerical methods increase sharply. Although numerous methods have been developed to date, the question of the effectiveness and accuracy of numerical methods remains open. Until now, different methods with uniform and non-uniform steps have been mainly used to solve singularly perturbed equations. As the value of the small parameter decreases, to increase the accuracy, it is necessary to refine the step of the difference grid. This, in turn, leads to a strong increase in the order of the matrix in the linear algebraic system being solved. Along with difference methods, spectral methods can be used to solve problems. In spectral methods, the solution to the equation is sought in the form of finite series in Chebyshev polynomials. The derivatives present in the equation are determined by differentiating the selected final series. When differentiating series, the order of the approximating polynomials is reduced, and this, in turn, affects the accuracy of the method used. In this paper, it is proposed to use the preliminary integration method to solve singularly perturbed equations. The essence of this method is as follows. The highest derivative and the right-hand side of the differential equation are expanded into finite series in Chebyshev polynomials of the first kind. Unlike spectral methods, in the preliminary integration method the highest derivative is expanded into a finite series. Before solving the problem, the series for the highest derivative is preliminarily integrated until an expression for solving the problem is found in the form of a finite series. When integrating series, unknown integration constants appear; they are determined from additional conditions of the problem. Only after this, the series for solving the derivatives of the right side are put into a singularly perturbed equation and a system of linear algebraic equations is obtained for determining the unknown expansion coefficients. It should be noted that when integrating series, the smoothness of the approximating polynomials improves, and this, in turn, increases the accuracy of the proposed method. At the same time, the order of the matrix of the algebraic system being solved does not increase. This ensures, at the same costs required in the spectral method, that the proposed method can solve a singularly perturbed equation even for small values of the small parameter of the problem. The high accuracy and efficiency of the preliminary integration method are demonstrated when solving a specific inhomogeneous singularly perturbed equation. The results of calculations are presented by comparing the approximate solution with the exact solution of the problem and with approximate solutions obtained by the spectral method.

***Keywords:*** *Inhomogeneous differential equation, boundary value problem, Chebyshev polynomials, preliminary integration, absolute error.*

## 1. Introduction

The construction of highly accurate and efficient methods for solving inhomogeneous singularly perturbed equations is an urgent problem in applied mathematics.

Let us briefly describe numerical methods aimed at solving singularly perturbed equations.

In [1], a finite-difference method for solving a singularly perturbed equation was proposed. The essence of this method is to replace the derivatives included in the equation with finite differences and solve the resulting system using linear algebra methods. Such a difference scheme requires a fairly fine grid step. When the value of the small parameter is on the order of $10^{-4}$, a uniform grid of 100 nodes is used to obtain sufficiently accurate ($10^{-3}$) results. In [2], in order to reduce the number of grid nodes, it is proposed to use a difference grid with a variable step. However, such a grid depends on several parameters, the choice of which encounters certain difficulties. A technique for constructing a non-uniform mesh for the numerical solution of a singularly perturbed equation was proposed in [3]. In [4], this method was used to solve the eigenvalue problem for an equation with a small parameter at the highest derivative, i.e., for the Orr–Sommerfeld equation. The results of the numerical solution of the Orr-Sommerfeld equation using a non-uniform mesh are

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presented in [5]. In [6], the results of constructing a non-uniform mesh for solving the Orr-Sommerfeld equation are presented, and the spectrum of eigenvalues for the Poiseuille flow is calculated. The numerical solution of the Orr-Sommerfeld equation using two-dimensional grids is presented in [7]. However, strict conditions are imposed on the parameters of such a grid to correctly describe the hydrodynamic properties of the flow. In [8], an inhomogeneous singularly perturbed second-order equation is solved by the spectral method. Numerical modeling of a fourth-order inhomogeneous singularly perturbed equation using the spectral method is presented in [9]. In the monograph [10], it is proposed to use the spectral grid method for numerical modeling of single-phase and two-phase flows. In this work, the convergence of the method is proved, and estimates of the speed of convergence of the method are obtained. The numerical solution of the Orr-Sommerfeld equation using the spectral-grid method is presented in [11]. The work shows the effectiveness and high accuracy of the proposed method. In [12], a spectral grid method is used to study the hydrodynamic stability of two-phase flows. In the two-phase flow under consideration, the dispersion (carrying) phase is gas, and the dispersed phase is solid particles. Numerical modeling of the Navier-Stokes equations in the system of a vortex and stream function using difference methods with a combination of Chebyshev polynomials of the first kind is presented in [13]. In [14], nonlinear waves with dissipation are numerically simulated by the spectral grid method. In [15], the convergence of the spectral-grid method was proved and estimates of the rate of convergence of the method were obtained for the Burgers equation with initial boundary conditions, where Chebyshev polynomials of the first kind were used. A review of methods for solving the problem of hydrodynamic stability is presented in [16]. In [17], the preliminary integration method was used to numerically simulate the eigenvalue problem of two-phase hydrodynamic flows.

A study of the cited literature and other sources shows that Chebyshev polynomials are widely used to study equations with a small parameter at the highest derivative. From the above review, it is clear that to solve the eigenvalue problem for a single-phase flow (Orr-Sommerfeld equation) and two-phase flow described by the eigenvalue problem for a system of nonlinear ordinary differential equations with a small parameter at the highest derivative, spectral and spectral-grid methods are successfully used.

From the above review, it is clear that works [1-7] are devoted to solving equations with a small parameter with the highest derivative using difference methods on uniform and non-uniform meshes. The following works are devoted to the use of spectral methods for the numerical solution of singularly perturbed equations of the second order [8] and fourth order [9]. In works [10-15], the spectral grid method was used for the numerical modeling of equations with a small parameter at the highest derivative. Numerical modeling of a singularly perturbed equation and a system of such equations using the preliminary integration method is presented in [16-17].

In difference methods, the derivatives included in the equation are replaced by finite differences, and the difference grid is constructed using a special transformation.

In spectral and spectral-grid methods, the solution to the equation is expanded into a finite series in Chebyshev polynomials. The derivatives present in a singularly perturbed equation are found by differentiating the selected finite series. It should be noted that when differentiating a series, the order of the approximating polynomials decreases (for example, with double differentiation, a polynomial of the fourth degree becomes a polynomial of the second degree), and this, in turn, affects the accuracy of the calculations. In the preliminary integration method, in contrast to spectral methods, not the solution of the equation, but the highest derivative is expanded into a finite series in Chebyshev polynomials. The lower derivatives and the solution to the singularly perturbed equation are found by preliminary integration of the series for the highest derivative. It should be noted that when integrating a finite series, the order of the approximating polynomials increases (for example, when integrating twice, a polynomial of the second degree becomes a polynomial of the fourth degree), the polynomials become smoother. We emphasize that both during differentiation and integration of a finite series, the order of the algebraic system being solved does not increase.

## 2. Problem Statement

In this work, to solve the problem posed in [4], the method of preliminary integration with polynomials is used. In the pre-integration method, singularity zones are not identified and do not depend on their location. The highest derivative of the differential equation and the right-hand side are expanded into a series of polynomials. By first integrating the series for the highest derivative, expressions for all lower derivatives and the desired solution are found in the form of series in polynomials. The integration constants that appear in this case are found from the conditions for satisfying the corresponding boundary conditions. Only after this the necessary series are put into the differential equation and a system of equations is obtained regarding the coefficients of the expansion of the series for the highest derivative. By solving the resulting system, the expansion coefficients are determined and placing them in the required series, it is possible to determine the values of the solution and its derivatives of any order, up to the highest derivative.

Let it be necessary to solve the following inhomogeneous singularly perturbed equation:

$$\varepsilon \frac{d^2u}{dy^2} + \frac{1}{2}\frac{du}{dy} = \frac{1}{8}\left(y+1\right), \quad y \in (-1,1), \tag{1}$$

with boundary conditions

$$u\left(-1\right) = u\left(+1\right) = 0, \tag{2}$$

where is $\varepsilon$ is a small parameter.

The trial function of the problem (1) and (2) has the form [4]:



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$$u(y)=\frac{\varepsilon-0,5}{1-e^{-1/\varepsilon}}\left(1-\varepsilon^{-(y+1)/2\varepsilon}\right)-\varepsilon\frac{y+1}{2}+\frac{(y+1)^2}{8}.$$

## 3. Solution Method

The highest derivative of the differential equation (1) and the right part of *f(y)* are searched for in the form of series:

$$\frac{d^2u}{dy^2}=\sum_{j=0}^{N}a_jT_j(y),\qquad f(y)=\sum_{i=0}^{N}b_iT_i(y),\tag{3}$$

where $T_j(y)$ are Chebyshev polynomials of the first kind.

After a two-time preliminary integration of the series (3), we have:

$$\frac{du}{dy}=\sum_{j=0}^{N+1}\sum_{i=0}^{N}f_{ji}^{(1)}a_iT_j(y)+C_1T_0(y),\tag{4}$$

$$u(y)=\sum_{j=0}^{N+2}\sum_{i=0}^{N}f_{ji}^{(0)}a_iT_j(y)+C_1T_1(y)+C_2T_0(y),\tag{5}$$

where $C_1, C_2$ are unknown integration constants. To determine them, we use the boundary conditions (2) and use the following properties of polynomials: $T_n(\pm1)=(\pm1)^n$.

Then, we have:

$$u(+1)=\sum_{j=0}^{N+2}\sum_{i=0}^{N}f_{ji}^{(0)}a_i+C_1+C_2=0,\tag{6}$$

$$u(-1)=\sum_{j=0}^{N+2}\sum_{i=0}^{N}(-1)^j f_{ji}^{(0)}a_i-C_1+C_2=0.\tag{7}$$

Adding equation (6) and (7), we get

$$u(+1)+u(-1)=\sum_{j=0}^{N+2}\sum_{i=0}^{N}f_{ji}^{(0)}a_i+\sum_{j=0}^{N+2}\sum_{i=0}^{N}(-1)^j f_{ji}^{(0)}a_i+2C_2=0.$$

From here, we define the constant $C_2$ as follows:

$$C_2=-\frac{1}{2}\sum_{i=0}^{N}\left[\sum_{j=0}^{N+2}f_{ji}^{(0)}+(-1)^j f_{ji}^{(0)}\right]a_i.$$

Similarly, subtracting equation (7) from equation (6), we determine the constant $C_1$ :

$$C_1=\frac{1}{2}\sum_{i=0}^{N}\left[\sum_{j=0}^{N+2}\left((-1)^j f_{ji}^{(0)}-\sum_{j=0}^{N+2}f_{ji}^{(0)}\right)\right]a_i,$$

We now introduce the following notation:

$$\delta_i^{(0)}=\sum_{j=0}^{N+2}f_{ji}^{(0)},\qquad \overline{\delta}_i^{(0)}=\sum_{j=0}^{N+2}(-1)^j f_{ji}^{(0)}.$$

Then, the expressions for constants $C_1$ and $C_2$ have the following form:

$$C_1=\frac{1}{2}\sum_{i=0}^{N}\left[\overline{\delta}_i^{(0)}+\delta_i^{(0)}\right]a_i,\tag{8}$$

$$C_2=-\frac{1}{2}\sum_{i=0}^{N}\left[\delta_i^{(0)}+\overline{\delta}_i^{(0)}\right]a_i.\tag{9}$$

Formulas (4) and (5), considering constants (7) and (9), are written in the following general form:

$$u^{(\beta)}(y)=\sum_{j=0}^{N+2-\beta}\sum_{i=0}^{N}g_{ji}^{(\beta)}a_iT_j(y),\qquad \beta=0,1,\tag{10}$$

where

$$g_{ji}^{(1)}=f_{ji}^{(1)}+\delta_{j,0}\frac{1}{2}\left(\overline{\delta}_i^{(0)}-\delta_i^{(0)}\right),\tag{11}$$

$$g_{ji}^{(0)}=f_{ji}^{(0)}+\delta_{j,1}\frac{1}{2}\left(\overline{\delta}_i^{(0)}-\delta_i^{(0)}\right)-\delta_{j,0}\frac{1}{2}\left(\delta_i^{(0)}-\overline{\delta}_i^{(0)}\right).\tag{12}$$

Here,

$\delta_{ij}=\begin{cases}1, if\ i=j,\\0, if\ i\neq j,\end{cases}$ denotes the Kronecker symbol.

Finally, substituting series (3) and (10) into (1) and equating the coefficients for the same degrees of polynomials, we obtain the following linear algebraic system for determining the expansion coefficients $a_{0,}a_1a_2...,a_n$ :

$$\sum_{k=0}^{n}\left[\varepsilon\delta_{ik}+\frac{1}{2}g_{ik}^{(1)}\right]a_k=b_i,\qquad i=0,1,2,...,n.\tag{13}$$

The right part of the system (13) is defined as follows:

It is known that

$$f(y)=\frac{1}{8}(y+1)=\sum_{i=0}^{N}b_iT_i(y).\tag{14}$$



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Hence, the coefficients in formula (14) are determined by the following inverse transformation [9-12]:

$$b_i = \frac{2}{Nc_i}\sum_{l=0}^{N}\frac{1}{c_l} f(y_l) T_i(y_l), \qquad i = 0,1,...,N,$$

or

$$b_i = \frac{2}{8Nc_i}\sum_{l=0}^{N}\frac{1}{c_l}(y_l+1) T_i(y_l), \qquad i = 0,1,...,N,$$

where $c_0 = c_N = 2,\ c_1 = 1,\ \ if\ \ l \neq 0; N,\ y_l = \cos\frac{\pi l}{N}$ is the collocation nodes for Chebyshev polynomials of the first kind.

Here is an algorithm for calculating constants [16-17]:

$$\delta_i^{(\beta)} = \sum_{j=0}^{N+2-\beta} f_{ji}^{(\beta)}, \qquad \bar{\delta}_i^{(\beta)} = \sum_{j=0}^{N+2-\beta} (-1)^j f_{ji}^{(\beta)},\ \ \beta = 0,1,$$

where

$$f_{ji}^{(1)} = \delta_{j,i+1}\beta_i^{(1)} + \delta_{j,i-1}\zeta_i^{(1)},$$
$$f_{ji}^{(0)} = \delta_{j,i+2}\beta_i^{(0)} + \delta_{j,i}\zeta_i^{(0)} + \delta_{j,i-2}\mu_i^{(0)}.$$

Here, constants $\beta, \zeta, \mu$ are calculated as follows:

$$\beta_i^{(1)} = \frac{c_i}{2(i+1)},\ i \geq 0, \quad \beta_i^{(0)} = \frac{\beta_i^{(1)}}{2(i+2)},\ \ i \geq 0,$$

$$\zeta_i^{(1)} = \frac{-1}{2(i-1)},\ i \geq 2, \quad \zeta_i^{(0)} = \frac{\zeta_i^{(1)} - \beta_i^{(1)}}{2i},\ \ i \geq 1,$$

$$\mu_i^0 = \frac{-\zeta_i^{(1)}}{2(i-2)},\ \ i \geq 3.$$

## 4. Discussion of The Results

Let us present the results of numerical calculations obtained by the preliminary integration method for solving the boundary value problem (1)-(3) when the value of a small parameter is for different numbers of polynomials. N=10, 20, 30,40 and 50.

Table 1 shows the results of the polynomials $y_l = \cos\frac{\pi l}{N}$ calculated in the nodes $l = 0,1,2,...,N$, when $N = 10$. For small numbers of polynomials, the influence of a small parameter on the dynamics of the numerical solution is observed. In this case, high-frequency-amplitude sawtooth jumps appear in the numerical solution. At the same time, it should be noted that the smaller the value of a small parameter $\varepsilon$, the more sawtooth jumps appear.

**Table 1.** Comparison of the trial function and numerical solution ($N = 10$)

| Nodes $Y_l$ on $l$ | $u_e$ - trial function | $u_a$ - numerical solution | $\Delta = \lvert u_e - u_a \rvert$ error |
|---|---|---|---|
| 3 | -0,4708209 | 0,47920272 | 0,9500237 |
| 5 | -0,37 | 0,5750956 | 0,9450956 |
| 7 | -0,1828062 | 0,7588535 | 0,9416597 |
| 9 | -0,0239276 | 0,8996691 | 0,9235967 |

From the results in Table 1, with a small number of polynomials, the trial function and numerical solutions are very different. These results are clearly presented in Figure 1.

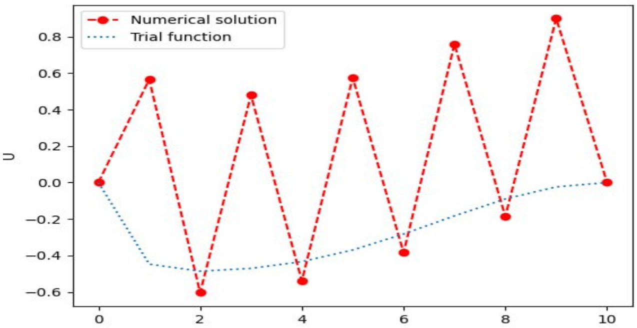


**Figure 1.** Dynamics of changes in the trial function and numerical solution ( $N = 10$ ).

From Figure 1, it can be seen that when approximating a solution with a small number of Chebyshev polynomials, sawtooth jumps of high amplitude appear.

Now, we gradually increase the number of approximating polynomials. In Table 2, the results obtained by the method of preliminary integration when $N = 20$ with a same value of parameter, $\varepsilon = 10^{-2}$ are presented.

**Table 2.** Comparison of trial function and numerical solutions

| Nodes $Y_l$ on $l$ | $u_e$ - trial function | $u_a$ - numerical solution | $\Delta = \lvert u_e - u_a \rvert$ error |
|---|---|---|---|
| 4 | -0,486361 | -0,488502 | 0,002141 |
| 8 | -0,433773 | -0,435351 | 0,001579 |
| 12 | -0,282354 | -0,283755 | 0,001401 |
| 16 | -0,089977 | -0,091128 | 0,001151 |

In Table 2, the numerical solution of the (1)-(3) is found with absolute accuracy $\Delta = 10^{-1}$. For clarity, the results of Table 2 are presented graphically in Figure 2.


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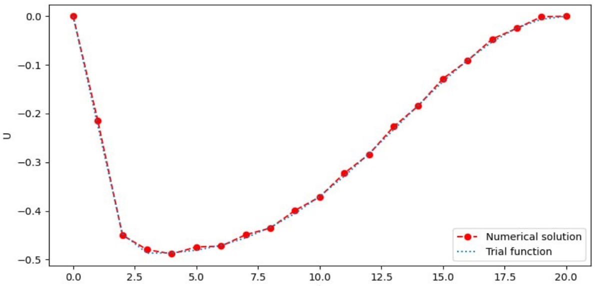


**Figure 2.** Dynamics of changes in the trial function and numerical solution ( $N = 20$ ).

In Table 2 and Figure 2, the pre-integration method provides very fast convergence of the numerical solution since the absolute error of the solution decreases sharply.

The resulting Table 3 shows the relationship between the maximum error and approximating polynomials at a value of a small parameter, $\varepsilon = 10^{-2}$.

**Table 3.** The relationship between the absolute error and polynomials.

| Number of polynomials, N | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
|---|---|---|---|---|---|---|---|---|
| Absolute error, $\Delta$ | $10^{-2}$ | $10^{-3}$ | $10^{-4}$ | $10^{-5}$ | $10^{-7}$ | $10^{-9}$ | $10^{-11}$ | $10^{-13}$ |

According to Table 3, with increasing polynomials, the absolute error decreases as a geometric progression. Now, we present the results of the calculation when the value of the small parameter is $\varepsilon = 10^{-2}$, i.e. 10 times less than the case discussed above. It should be noted that almost all of the above methods become unsuitable for studying the dynamics of changes in the solution of the problem (1)-(2) with this small parameter value of $\varepsilon = 10^{-3}$. In this case, as noted above, high-frequency-amplitude sawtooth jumps are clearly manifested in the solution area. In Table 4, the results are given when the number of polynomials is equal to $N = 40$, $\varepsilon = 10^{-3}$.

**Table 4.** Comparison of the trial function and numerical solution ( $N = 40$ ).

| Nodes $Y_l$ on $l$ | $u_e$ - trial function | $u_a$-numerical solution | $\Delta = \lvert u_e - u_a$ - error |
|---|---|---|---|
| 15 | -0,451674 | 0,702899 | 1,154572 |
| 25 | -0,260715 | 0,893573 | 1,154287 |
| 35 | -0,037298 | 1,115398 | 1,152696 |

It can be seen that with the value of the small parameter of $\varepsilon = 10^{-3}$, the numerical solution is very different from the trial function. This is explained by the fact that the number of approximating polynomials is not enough to display the dynamics of changes in the solution of the problem.

The results given in Table 4 will be represented graphically in Figure 3.

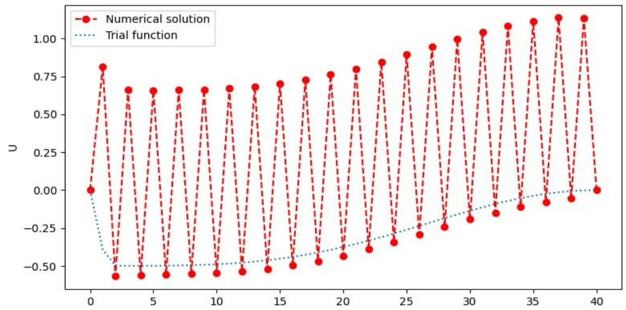


**Figure 3.** Dynamics of changes in the trial function and numerical solution ( $N = 40$ ).

It can be seen that the frequency and amplitude of the sawtooth are too high.

In Table 5, the results of the polynomials are $N = 60$, $\varepsilon = 10^{-3}$.

**Table 5**. Comparison of the trial function and numerical solution ( $N = 60$ ).

| Nodes $Y_l$ on $l$ | $u_e$ - trial function | $u_a$ - numerical solution | $\Delta = \lvert u_e - u_a \rvert$ error |
|---|---|---|---|
| 10 | -0,4968 | -0,5010 | 0,0042 |
| 20 | -0,4680 | -0,4720 | 0,0040 |
| 30 | -0,3745 | -0,3785 | 0,0039 |
| 40 | -0,2225 | -0,2225 | 0,0040 |
| 50 | -0,0647 | -0,0685 | 0,0038 |

The results in Table 5 are clearly illustrated in Figure 4.

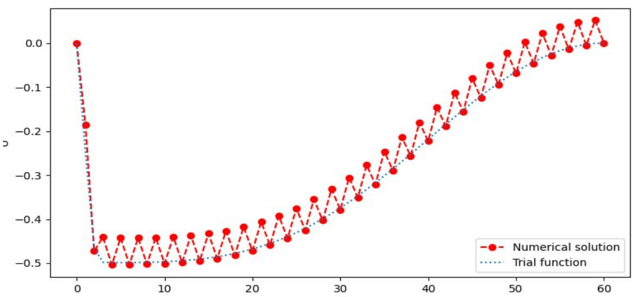


**Figure 4.** Dynamics of changes in the trial function and numerical solution ( $N = 60$ ).

In Figure 4, the amplitude of the sawtooth jumps is significantly small, and the maximum error is of the order of $\Delta = 10^{-1}$.

The results of comparing the trial function and numerical solution when $N = 100$ and $\varepsilon = 10^{-3}$ are shown in Table 6.



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**Table 6.** Comparison of trial function and numerical solutions ( $N=100$ )

| Nodes $Y_l$ at $l$ | $u_e$ -trial function | $u_a$-numerical solution | $\Delta=\|u_e-u_a\|$ Error |
|---|---|---|---|
| 10 | 0,498725 | -0,498729 | 4,14*$10^{-6}$ |
| 20 | 0,494536 | -0,494539 | 3,30*$10^{-6}$ |
| 30 | 0,477966 | -0,477969 | 3,09*$10^{-6}$ |
| 40 | 0,439663 | -0,439667 | 3,00*$10^{-6}$ |
| 50 | 0,374500 | -0,374503 | 2,96*$10^{-6}$ |
| 60 | 0,285464 | -0,285467 | 2,92*$10^{-6}$ |
| 70 | 0,184661 | -0,184664 | 2,86*$10^{-6}$ |
| 80 | 0,090837 | -0,090839 | 2,72*$10^{-6}$ |
| 90 | 0,024148 | -0,024150 | 2,18*$10^{-6}$ |

Graphical representations of the results of calculations given in Table 6 are shown in Figure 5.

According to Figure 5, the numerical solution practically coincides with the trial function of problem (1)-(2) and with the small parameter $\varepsilon=10^{-3}$. The maximum error will be of the order of $\Delta=10^{-5}$.

Here, Table 7 shows th establishing relationship between the error and approximating Chebyshev polynomials at a small parameter value of $\varepsilon=10^{-3}$.

**Table 7.** The relationship between the error and polynomials ( $\varepsilon=10^{-3}$ )

| Number of polynomials, N | 60 | 70 | 80 | 90 | 100 |
|---|---|---|---|---|---|
| Error, $\Delta$ | $10^{-1}$ | $10^{-2}$ | $10^{-3}$ | $10^{-4}$ | $10^{-5}$ |

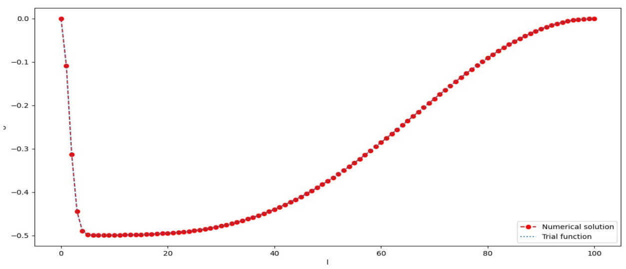


**Figure 5.** Dynamics of changes in the trial function and numerical solution ( $N=100$ ).

From Table 7, with an increase in the number of approximating Chebyshev polynomials, the maximum absolute error decreases with the rate of geometric progression and, at the same time, the value of a small parameter. All numerical results were obtained using a Python program.

The main criterion for assessing the effectiveness of an arbitrary numerical method is the number of arithmetic operations. When approximating a singularly perturbed equation by difference and spectral or by the preliminary integration method, a system of linear algebraic equations is obtained. The order of the matrix in a linear algebraic system depends on the number of difference grid nodes (N) or the number of polynomials used in finite Chebyshev series in the spectral method or the preliminary integration method (N). Let us assume that the resulting algebraic system is solved by the Gaussian method. It is known that the formula calculates the number of arithmetic operations in the Gauss method

$$Q=\frac{2}{3}N^3.$$

Let us compare the effectiveness of the methods used to solve a singularly perturbed equation in terms of the number of arithmetic operations and accuracy.

We present the results in Table 8.

**Table 8.** Comparison of methods in terms of efficiency and accuracy at $\varepsilon=10^{-2}$.

| Method | Number N | Number of arithmetic operations Q | Maximum absolute error $\Delta$ |
|---|---|---|---|
| Finite-difference [3] | 20 | 5333 | 0.1195 |
| | 50 | 83333 | 0.0521 |
| Spectral [8] | 10 | 666 | 0.97 |
| | 20 | 5333 | 0.0027 |
| | 50 | 83333 | $10^{-10}$ |
| Pre-integration method | 10 | 666 | 0.95 |
| | 20 | 5333 | 0.0021 |
| | 50 | 83333 | $10^{-13}$ |

From the results given in Table 8 it is clear that the preliminary integration method has high accuracy and efficiency. Thus, the pre-integration method is a universal and reliable mathematical tool for solving a singularly perturbed equation.

## 5. Conclusion

i. For the numerical solution of an inhomogeneous singularly perturbed equation, a new high-precision and efficient method is proposed - the method of preliminary integration.
ii. The inhomogeneous singularly perturbed equation is solved by the proposed method for various values of the small parameter of the problem.
iii. Comparison of the obtained results with the exact solution of the problem and the approximate solution obtained by the spectral method shows the high accuracy and efficiency of the preliminary integration method.
iv. Tabular and graphical results illustrating the accuracy and efficiency of the method are presented.


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Malaysian Journal of Science 44(1): 63-69 (March 2025)

## 6. References


Abutaliev F. B., Normurodov Ch. B Mathematical modeling of the problem of hydrodynamic stability, Tashkent:Fan va texnologiya:2011:188

Bakhvalov K.S (1969) Towards optimization of methods for solving boundary value problems in the presence of a boundary layer, 4(9): 841-859.

Krylov A.A., Malykhina I.D (1968) Solving the eigenvalue problem for the Orr-Sommerfeld equation using the difference method. Comput. methods and programming. 11:44-54.

Liseikin V. D., Yanenko N. N (1981) On a uniformly convergent algorithm for the numerical solution of a second-order ordinary differential equation with a small parameter at the highest derivative, Numerical methods in continuum mechanics,12:45–56.

Loer St (1969) Examination of the stability of disturbed boundary- layer flow by a numerical method, Phys. fluids. 12(12):139-143

Normurodov Ch. B (2005) On one effective method for solving the Orr-Sommerfeld equation, Mathematical Modeling-Moscow, 9(17):35-42.

Normurodov Ch. B (2007) Mathematical modeling of hydrodynamic problems for two-phase plane-parallel flows, Mathematical Modeling-Moscow, 6(19): 53-60.

Normurodov Ch. B., Abdurakhimov B. F., Djurayeva N. T., Viswanathan K. K., Saravanan D (2023) Application of the Preliminary Integration Method for Numerical Simulation of Two-Phase Hydrodynamic Flows, European Chemical Bulletin, 12: 959-968. ISSN 2063-5346. Open Access E-journal.

Normurodov Ch. B., Djurayeva N. T (2022) Review of methods for solving the problem of hydrodynamic stability. Problems of Computational and Applied mathematics, 1(38):83-90

Normurodov Ch. B., Kholiyarov Ch. E., Gulomkodirov K. A., Djurayeva N. T (2022) Numerical simulation of the inverse problem for the vortex-current equation. AIP Conf. Proc., 2637:040018. http://dx.doi.org/10.1063/5.0118605.

Normurodov Ch. B., Tilovov M.A., Tursunova B.A., Djurayeva N.T (2023) Numerical modeling of inhomogeneous singularly perturbed fourth-order boundary value problems using the spectral method. Problems of Computational and Applied mathematics, 5(52): 83-90

Normurodov Ch. B., Toyirov A. X., Yuldashev Sh. M (2022) Numerical modeling of a wave in a nonlinear medium with dissipation. AIP Conf. Proc., 2637:040005, http://dx.doi.org/10.1063/5.0118459.

Normurodov Ch.B., Toyirov A. Kh., Ziyakulova Sh.A., Viswanathan K.K (2024) Convergence of Spectral- Grid Method for Burgers Equation with Initial- Boundary Conditions. Mathematics and Statistics, 12(2):115-125. DOI: 10.13189/ms.2024.120201.

Normurodov Ch. B., Tursunova B. A (2023) Numerical modeling of the boundary value problem of an ordinary differential equation with a small parameter at the highest derivative by Chebyshev polynomials of the second kind, Results in Applied Mathematics, 19:1-5.

Thomas H.H (1953) The stability of plane Poiseuille flow, Phys.rev, 4(91):780-783.

Zharilkasinov A., Liseykin V.D., Skobelev B.Yu., Yanenko N.N (1983) Application of a non-uniform grid for the Orr-Sommerfeld numerical solution, Numerical methods of continuum mechanics. 5(14): 45-54.

Zharilkasinov A., Skobelev B.Yu., Yanenko N.N (1984) Effective non-uniform mesh for the Orr-Sommerfeld equation and Poiseuille flow spectrum. (Preprint/RAN.Sib.department. Institute of Theor. And Applied Mechanics), 21: 35.




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