Stability Analysis and Maximum Profit of One Prey-Two Predators Model under Constant Effort of Harvesting

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ABSTRACT   In this paper we present a deterministic and continuous model for one prey–two predators population model based on the Lotka-Volterra model. The two predators are subjected to constant effort of harvesting. We study analytically the necessary conditions of harvesting to ensure existence of the equilibrium points and their stabilities. The methods used to analyse the stability are linearisation and Hurwitz stability test. The results show that there is an asymptotically stable equilibrium point in positive octane for the model without constant effort of harvesting. We found that there is an asymptotically stable equilibrium point in positive octane for the model with constant effort of harvesting. The stable equilibrium point for the model with constant effort of harvesting is then related to profit function which we found to have maximum profit. This means that the prey and predator populations can live in coexistence and give maximum profit although the two predators are harvested with constant effort of harvesting.

ABSTRAK   Dalam kertas ini kita kemukakan suatu model berketentuan dan selanjar untuk suatu model populasi satu mangsa – dua pemangsa berdasarkan model Lotka-Volterra. Kedua – dua pemangsa adalah tertakluk kepada tuaian usaha malar. Kita mengkaji secara analitik syarat perlu bagi tuaian untuk menjamin kewujudan titik keseimbangan dan kestabilannya. Kaedah yang digunakan untuk menganalisis kestabilan adalah kaedah pelinearan dan ujian kestabilan Hurwitz. Keputusannya menunjukkan bahawa wujud suatu titik keseimbangan yang stabil secara asimptot dalam oktan positif untuk model tanpa tuaian usaha malar. Kita dapati bahawa wujud suatu titik keseimbangan yang stabil secara asimptot dalam oktan positif untuki model dengan tuaian usaha malar. Titik keseimbangan yang stabil bagi model dengan tuaian usaha malar ini kemudian dihubungkan dengan fungsi keuntungan dan didapati mempunyai keuntungan maksimum. Ini bermakna bahawa populasi mangsa dan pemangsa dapat hidup bersama dan memberi keuntungant maksimum walaupun kedua – dua pemangsa dituai dengan tuaian usaha malar.

(Prey-predator, Hurwitz Stability Test, harvesting, profit)

INTRODUCTION

A number of researchers have studied and attempted to find more information on model involving three or more compartments. A two prey-one predator system based on Lotka-Volterra model has been analysed by Takeuchi and Adachi [1]. The result indicates that stable coexistence of three species at equilibrium point is possible only when the predator prefers the dominant prey. If the predator prefers the inferior prey, the prey inevitably becomes extinct. A three-dimensional model that presents a model of the prey population in two habitats including the dynamics of the predator population has been proposed and analysed by Freedman [2]. The case of no harvesting is analysed and a criterion for persistence is given.

Ebenhoh [3] has studied one prey-two predator models where the two predators compete for the prey. Coexistence is possible if both one prey-one predator boundary systems show predator prey limit cycle. Invasion of the other predator is then possible in both cases. If the parameter combinations are changed to damp the limit cycle, coexistence becomes impossible. Another one prey-two predator model has been considered by Cale and O'Neill [4]. The model was simulated with selected initial values. The selected initial values are to cover the region of phase space and the result indicates that all simulations are run to steady state.

Rinaldi and Muratori [5] have studied one prey-two predator model (prey, predator, and super-predator). The analysis of the geometry shows that the state of the system tends toward a stable limit cycle. In particular, for suitable values of the parameters, the populations can periodically appear during a fraction of a low frequency limit cycle. A general model, including three-dimensional ones where two predators compete for a single prey, has been considered by Farkas and Freedman [6]. An example of the model shows that the equilibrium point, which is in the positive octane, is asymptotically stable for suitable values of the parameters.

In this paper we present a deterministic, continuous model for one prey–two predator population based on Lotka-Volterra model. The predators are subjected to constant effort of harvesting. Existence of the equilibrium points is investigated. The stable equilibrium point is then related to the maximum profit function.

ONE PREY-TWO PREDATORS MODEL

Consider a model which involves three populations, one prey and two predators, based on Lotka-Volterra prey-predator model. In this model, we assume that the two predators have no interactions. They compete for a single prey. The model is

$$\dot{x} = x\left(a - bx - \alpha y - \beta z\right)$$
$$\dot{y} = y\left(-c - ey + \zeta x\right) \qquad (1)$$
$$\dot{z} = z\left(-f - gz + \delta x\right)$$

where x, y and z represent the number of the prey, the first predator, and the second predator, respectively, at time t. Here a, b, c, e, f, g, α, β, δ, and ζ are all positive parameters. This model considers that in the absence of predators, the number of prey grows following logistic growth. In the absence of a prey population, the numbers of the two predators decrease exponentially and then tend to zero. Such growth rate of predator population of model (1) has been considered by Kuang and Takeuchi [7] and Liu and Wang [8]. The authors considered the stability of two preys-one predator model with diffusion.

The equilibrium points of prey-predator model (1) are

$$E1 = \left(0, 0, 0\right), \quad E2 = \left(a/b, 0, 0\right),$$

$$E3 = \left(\frac{ag + \beta f}{bg + \beta\delta}, 0, \frac{a\delta - bf}{bg + \beta\delta}\right),$$

$$E4 = \left(\frac{ae + \alpha c}{be + \alpha\zeta}, \frac{a\zeta - bc}{be + \alpha\zeta}, 0\right),$$

and

$$E5 = (x^*, y^*, z^*)$$

where

$$x^* = \frac{aeg + \alpha cg + \beta ef}{beg + \alpha\zeta g + \beta\delta e},$$

$$y^* = \frac{a\zeta g + \beta\zeta f - bcg - \beta c\delta}{beg + \alpha\zeta g + \beta\delta e},$$

and $\quad z^* = \frac{a\delta e + \alpha c\delta - bef - \alpha f\zeta}{beg + \alpha\zeta g + \beta\delta e}.$

To study the local behavior of model (1), firstly we discuss the stability of the equilibrium points in the model by using linearisation method. Secondly, we follow Hurwitz stability test to investigate the stability of the equilibrium point $E5$.

Now let us investigate the stability of model (1) at equilibrium points $E1$, $E2$, $E3$ and $E4$. The Jacobian matrix of model (1) takes the form

$$J = \begin{pmatrix} a-2bx-\alpha y-\beta z & -\alpha x & -\beta x \\ \zeta y & -c-2ey+\zeta x & 0 \\ \delta z & 0 & -f-2gz+\delta x \end{pmatrix}.$$

$$(2)$$

At equilibrium point $E1$, we have

$$J_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & -f \end{pmatrix}.$$

From the Jacobian matrix we have $\det(J_1) = acf > 0$. The eigenvalues of the Jacobian matrix J_1 are $r_1 = a$, $r_2 = -c$ and $r_3 = -f$. Since the Jacobian matrix J_1 has negative and positive eigenvalues, then the equilibrium point $E1$ is an unstable saddle point. At equilibrium point $E2$, we have

$$J_2 = \begin{pmatrix} -a & -a\alpha/b & -a\beta/b \\ 0 & -(bc+a\zeta)/b & 0 \\ 0 & 0 & -(bf+a\delta)/b \end{pmatrix}.$$

From the Jacobian matrix we have $\det(J_2) = -a(bc + a\zeta)(bf + a\delta)/b^2$. The eigenvalues of the Jacobian matrix J_2 are $r_1 = -a$, $r_2 = \dfrac{a\zeta - bc}{b}$, and $r_3 = \dfrac{a\delta - bf}{b}$. If $a\zeta - bc < 0$ and $a\delta - bf < 0$, then the eigenvalues r_2 and r_3 are both negative. Then the equilibrium point $E2$ is asymptotically stable. While if $a\zeta - bc > 0$ or $a\delta - bf > 0$, then at least one of the eigenvalues r_2 or r_3 is positive. Then the equilibrium point is an unstable saddle point.

At equilibrium point $E3$, we have

$$J_3 = \begin{pmatrix} A_3 & B_3 & C_3 \\ 0 & E_3 & 0 \\ F_3 & 0 & G_3 \end{pmatrix},$$

where

$$A_3 = -\frac{b(ag + \beta f)}{bg + \beta\delta}, B_3 = -\frac{\alpha(ag + \beta f)}{bg + \beta\delta},$$

$$C_3 = -\frac{\beta(ag + \beta f)}{bg + \beta\delta}, E_3 = \left(\frac{a\zeta g + \beta\zeta f - bcg - \beta c\delta}{bg + \beta\delta}\right),$$

$$F_3 = \frac{\delta(a\delta - bf)}{bg + \beta\delta}, \text{ and } G_3 = -\frac{g(a\delta - bf)}{bg + \beta\delta}. \tag{3}$$

From the Jacobian matrix we have $\det(J_3) = E_3(A_3 G_3 - C_3 F_3) > 0$. From (3) we see that $A_3 < 0$, $B_3 < 0$, $C_3 < 0$, $F_3 > 0$, and $G_3 < 0$. The eigenvalues of the Jacobian matrix J_3 are

$$r_1 = E_3$$

and

$$r_{2,3} = \frac{(A_3 + G_3)}{2} \pm \frac{\sqrt{(A_3 + G_3)^2 - 4(A_3 G_3 - C_3 F_3)}}{2}.$$

From the conditions of element of the Jacobian matrix J_3, it follows that both the eigenvalues r_2 and r_3 are either negative or conjugate complex with negative real part. If $a\zeta g + \beta\zeta f - bcg - \beta c\delta > 0$, then E_3 becomes positive, thus $r_1 > 0$. Therefore the equilibrium point $E3$ is unstable. If $a\zeta g + \beta\zeta f - bcg - \beta c\delta < 0$ then the equilibrium point $E3$ is asymptotically stable.

At equilibrium point $E4$, we have

$$J_4 = \begin{pmatrix} A_4 & B_4 & C_4 \\ \Delta_4 & E_4 & 0 \\ 0 & 0 & G_4 \end{pmatrix},$$

where

$$A_4 = -\frac{b(ae + \alpha c)}{be + \alpha\zeta}, B_4 = -\frac{\alpha(ae + \alpha c)}{be + \alpha\zeta},$$

$$C_4 = -\frac{\beta(ae + \alpha c)}{be + \alpha\zeta}, \Delta_4 = \frac{\zeta(a\zeta - bc)}{be + \alpha\zeta}, \tag{4}$$

$$E_4 = -\frac{e(a\zeta - bc)}{be + \alpha\zeta}, \text{ and}$$

$$G_4 = \left(\frac{a\delta e + \alpha c\delta - \alpha f\zeta - bef}{be + \alpha\zeta}\right).$$

It is easy to see that $\det(J_4) = G_4(A_4 E_4 - B_4\Delta_4)$. We can also see in (4) that $A_4 < 0$, $B_4 < 0$, $C_4 < 0$, $\Delta_4 > 0$, and $E_4 < 0$. Then the eigenvalues under this situation are $r_1 = G_4$ and

$$r_{2,3} = \frac{(A_4 + E_4)}{2} \pm \frac{\sqrt{(A_4 + E_4)^2 - 4(A_4 E_4 + B_4\Delta_4)}}{2}$$

From the conditions of elements of the Jacobian matrix J_4, we know that both the eigenvalues r_2 and r_3 are either negative or conjugate complex with negative real part. If $a\delta e + \alpha c\delta - \alpha f\zeta - bef > 0$, then G_4 becomes positive or, equivalently, $r_1 > 0$. Therefore the equilibrium point $E4$ is unstable. If

$a\delta e + \alpha c \delta - \alpha f \zeta - bef < 0$ then the equilibrium point $E4$ is asymptotically stable.

Theorem 1

Let $E5$ be the equilibrium point of model (1) in the positive octane. Then it is qualitatively stable and an attractor trajectory.

Proof. Substituting the equilibrium point $E5$ into (2) gives the Jacobian matrix J_s as follows

$$J_s = \begin{pmatrix} A_s & B_s & C_s \\ \Delta_s & E_s & 0 \\ F_s & 0 & G_s \end{pmatrix},$$

where

$$A_5 = -\frac{b\left(aeg + \alpha cg + \beta ef\right)}{beg + \alpha \zeta g + \beta d\delta},$$

$$B_5 = -\frac{\alpha\left(aeg + \alpha cg + \beta ef\right)}{beg + \alpha \zeta g + \beta e\delta},$$

$$C_5 = -\frac{\beta\left(aeg + \alpha cg + \beta ef\right)}{beg + \alpha \zeta g + \beta e\delta},$$

$$\Delta_5 = \frac{\zeta\left(a\zeta g + \beta \zeta f - bcg - \beta c\delta\right)}{beg + \alpha \zeta g + \beta e\delta},$$

$$E_5 = -\frac{\zeta\left(a\zeta g + \beta \zeta f - cbg - c\beta \delta\right)}{beg + \alpha \zeta g + \beta \delta e},$$

$$F_5 = \frac{\delta\left(a\delta e + \alpha c\delta - bef - \alpha \zeta f\right)}{beg + \alpha \zeta g + \beta \delta e},$$

and $G_5 = -\frac{g\left(a\delta e + \alpha c\delta - bef - \alpha \zeta f\right)}{beg + \alpha \zeta g + \beta \delta e}.$

(5)

Since the equilibrium point $E5$ is in the positive octane, so the conditions

$$a\zeta g + \beta \zeta f - bcg - \beta c\delta > 0$$

and $\quad a\delta e + \alpha c\delta - bef - \alpha \zeta f > 0$

are satisfied. It follows, from (5), that $A_s < 0$, $B_s < 0$, $C_s < 0$, $\Delta_s > 0$, $E_s < 0$, $F_s > 0$, and $G_s < 0$. Then the Jacobian matrix together with its properties satisfy all of the qualitative stability conditions, Jeffries [9], i.e., (i) $A_s < 0$, $E_s < 0$, $G_s < 0$, (ii) $A_s \neq 0$, $E_s \neq 0$, $G_s \neq 0$, (iii) $B_5 \Delta_s < 0$, $C_s F_s < 0$, (iv) $B_5 0 F_5 = 0$, $C_5 0 \Delta_5 = 0$, and (v) $\det\left(J_5\right) = \left(A_5 E_5 G_5 - B_5 \Delta_5 G_5 - C_5 E_5 F_5\right) < 0$. Further, we conclude that the equilibrium point $E5$ is qualitatively stable. This means that the equilibrium point $E5$ is stable.

We would also like to investigate the kind of stability of the equilibrium point $E5$ using Hurwitz stability test, see Jeffries [10] and Willems [11]. The characteristic equation of the Jacobian matrix J_s is

$$f(r) = r^3 - \left(A_5 + E_5 + G_5\right)r^2 + \left(A_5 E_5 + A_5 G_5 - B_5 \Delta_5\right.$$
$$\left. - C_5 F_5 + E_5 G_5\right)r - A_5 E_5 G_5 + B_5 \Delta_5 G_5 + C_5 E_5 F.$$

From the characteristic equation we have

$$p_0 = \begin{pmatrix} -A_5 E_5 G_5 + B_5 \Delta_5 G_5 \\ + C_5 E_5 F_5 \end{pmatrix}; \quad p_0 > 0,$$

$$p_1 = \begin{pmatrix} A_5 E_5 + A_5 G_5 - B_5 \Delta_5 \\ - C_5 F_5 + E_5 G_5 \end{pmatrix}; \quad p_1 > 0,$$

$$p_2 = -\left(A_5 + E_5 + G_5\right); \quad p_2 > 0, \text{ and}$$

$$p_2 p_1 - p_0 = A_5 B_5 \Delta_5 - A_5^2 E_5 - A_5^2 G_5$$
$$+ A_5 C_5 F_5 - 2 A_5 E_5 G_5 - A_5 G_5^2$$
$$+ C_5 F_5 G_5 - E_5 G_5^2 + B_5 \Delta_5 E_5$$
$$- A_5 E_5^2 - E_5^2 G_5; \quad p_2 p_1 - p_0 > 0.$$

(6)

From (6) we know that the Hurwitz stability test is satisfied, so the equilibrium point $E5$ is an attractor trajectory. It is an asymptotically stable equilibrium point.

When the equilibrium point $E5$ is in the positive octane, the equilibrium points $E1$, $E2$, $E3$, and $E4$ may also exist, i.e., none of the components of $E1$, $E2$, $E3$, and $E4$ has negative value. If the equilibrium points $E1$, $E2$, $E3$, and $E4$ exist, then it is impossible that they are stable equilibrium points. The equilibrium points $E1$ and $E2$ are both unstable saddle points whereas the equilibrium points $E3$ and $E4$ are both not attractor trajectory and not asymptotically stable equilibrium points.

Example 1.

Consider the model (1) with parameters $a = 2$, $b = 0.0001$, $\alpha = 0.003$, $c = 0.05$, $e = 0.2$, $f = 0.04$, $g = 0.1$, $\beta = 0.004$, $\zeta = 0.002$, and $\delta = 0.001$. The five equilibrium points and their stabilities are given in Table 1.

Table 1. Equilibrium points and their stabilities

	Equilibrium point	Eigenvalues	Stability
E_1	$(0, 0, 0)$	-0.0500, -0.0400, 2.0000	Unstable saddle point
E_2	$(20000, 0, 0)$	-2.0000, 39.9500, 19.9600	Unstable saddle point
E_3	$(14297.1429, 0, 142.5714)$	-2.1004, -13.5864, 28.5443	Unstable saddle point
E_4	$(15390.3846, 153.6538, 0)$	-2.0335, -30.2363, 15.3504	Unstable saddle point
E_5	$(11778.5394, 117.5353, 117.3853)$	-2.1430, -11.1434, -23.1370	Asymptotically stable

MODEL WITH CONSTANT EFFORTS OF HARVESTING

We consider a constant effort of harvesting in model (1) where the two predator populations are subjected to constant effort of harvesting. The rate of harvesting is proportional to the size of the respective predator population. Here, we assume that the two predators are economically valuable and hence the predators are harvested. Model (1) becomes

$$\dot{x} = x\left(a - bx - \alpha y - \beta z\right)$$
$$\dot{y} = y\left(-c - ey + \zeta x\right) - E_y y \qquad (7)$$
$$\dot{z} = z\left(-f - gz + \delta x\right) - E_z z$$

where E_y and E_z are constant efforts of harvesting for the population y and z, respectively. By letting $m = c + E_y$ and $n = f + E_z$, model (7) can be rewritten in the form

$$\dot{x} = x\left(a - bx - \alpha y - \beta z\right)$$
$$\dot{y} = y\left(-m - ey + \zeta x\right) \qquad (8)$$
$$\dot{z} = z\left(-n - gz + \delta x\right)$$

where m and n are also positive constants.

The equilibrium points of model (8) are, namely, $E_H 1 = (0, 0, 0)$, $E_H 2 = (a/b, 0, 0)$,

$$E_H 3 = \left(\frac{ag + \beta n}{bg + \beta \delta}, 0, \frac{a\delta - bn}{bg + \beta \delta}\right),$$

$$E_H 4 = \left(\frac{ae + \alpha m}{be + \alpha \zeta}, \frac{a\zeta - bm}{be + \alpha \zeta}, 0\right),$$

and $\quad E_H 5 = (x_*, y_*, z_*)$,

where $\quad x_* = \dfrac{aeg + \alpha mg + \beta en}{beg + \alpha \zeta g + \beta \delta e}$,

$$y_* = \frac{a\zeta g + \beta \zeta n - bmg - \beta m\delta}{beg + \alpha \zeta g + \beta \delta e},$$

and $\quad z_* = \dfrac{ae\delta + \alpha m\delta - ben - \alpha n\zeta}{beg + \alpha \zeta g + \beta \delta e}$.

The Jacobian matrix of model (8) is

$$J_H = \begin{pmatrix} A_H & B_H & C_H \\ \Delta & E_H & 0 \\ F_H & 0 & G_H \end{pmatrix}, \qquad (9)$$

where $\quad A_H = a - 2bx - \alpha y - \beta z$, $\quad B_H = -\alpha x$, $C_H = -\beta x$, $\quad \Delta_H = \zeta y$, $\quad E_H = -m - 2ey + \zeta x$, $F_H = \delta z$, and $G_H = -n - 2gz + \delta x$.

Models (1) and (8) are mathematically similar, so the analyses of stability of their equilibrium points are similar. At the equilibrium point $E_H 1$, the eigenvalues of the Jacobian matrix J_H associated with this equilibrium point are $r_1 = a$, $r_2 = -m$, and $r_3 = -n$. Then the equilibrium point $E_H 1$ is an unstable saddle point.

If $E_y > (a\zeta - bc)/b$ and $E_z > (a\delta - bf)/b$, then the equilibrium point $E_H 2$ is asymptotically stable. If the equilibrium point $E_H 3$ exists and model (8) satisfies the conditions

$$a\delta - bf > 0, \quad a\zeta g + \beta \zeta f - bcg - \beta c\delta > 0,$$

$$(bg + \beta \delta) E_y - \beta \zeta E_z > a\zeta g + \beta \zeta f - bcg - \beta c\delta,$$

and $0 < E_z < (a\delta - bf)/b$, then the equilibrium point $E_H 3$ is asymptotically stable. If the conditions

$$a\zeta - bc > 0, \quad a\delta e + \alpha c\delta - \alpha f\zeta - bef > 0,$$

$-\alpha\delta E_y + (be + \alpha\zeta) E_z > a\delta e + \alpha c\delta - \alpha f\zeta - bef$,
and $0 < E_y < (a\zeta - bc)/b$ are satisfied, then the equilibrium point $E_H 4$ is asymptotically stable. However, when the two predators are subjected to constant effort of harvesting and the equilibrium points $E_H 2$, $E_H 3$, or $E_H 4$ are stable, then at least one of the two predators will become extinct.

Theorem 2.

Let $E_H 5$ be one of the equilibrium points of model (8) which occurs in the positive octane. Then the equilibrium point $E_H 5$ is qualitatively stable and is an attractor trajectory.

Proof. Substituting the equilibrium point $E_H 5$ into (9) gives the Jacobian matrix J_{H5} as follows

$$J_{H5} = \begin{pmatrix} A_{H5} & B_{H5} & C_{H5} \\ \Delta_{H5} & E_{H5} & 0 \\ F_{H5} & 0 & G_{H5} \end{pmatrix},$$

where

$$
\begin{aligned}
A_{H5} &= -\frac{b(aeg + amg + \beta en)}{beg + \alpha\zeta g + \beta\delta e}, \\
B_{H5} &= -\frac{\alpha(aeg + amg + \beta en)}{beg + \alpha\zeta g + \beta\delta e}, \\
C_{H5} &= -\frac{\beta(aeg + amg + \beta en)}{beg + \alpha\zeta g + \beta\delta e}, \\
\Delta_{H5} &= \frac{\zeta(a\zeta g + \beta\zeta n - bmg - m\beta\delta)}{beg + \alpha\zeta g + \beta\delta e}, \\
E_{H5} &= -\frac{\zeta(a\zeta g + \beta\zeta n - bmg - m\beta\delta)}{beg + \alpha\zeta g + \beta\delta e}, \\
F_{H5} &= \frac{\delta(a\delta e + am\delta - ben - \alpha\zeta n)}{beg + \alpha\zeta g + \beta\delta e},
\end{aligned}
\tag{10}
$$

and

$$G_{H5} = -\frac{g(a\delta e + am\delta - ben - \alpha\zeta n)}{beg + \alpha\zeta g + \beta\delta e}.$$

From the Jacobian matrix we have $\det(J_{H5}) = A_{H5}E_{H5}G_{H5} - B_{H5}\Delta_{H5}G_{H5} - C_{H5}E_{H5}F_{H5}$. Since the equilibrium point $E_H 5$ is in the positive octane, the conditions

$$a\zeta g + \beta\zeta n - bmg - m\beta\delta > 0$$

and

$$a\delta e + am\delta - ben - \alpha\zeta n > 0$$

are satisfied. These conditions are equivalent to say, $(E_y, E_z) \in R$, where

$$
\begin{aligned}
R = \Big\{ &(E_y, E_z) \mid (bg + \beta\delta)E_y - \beta\zeta E_z \\
&< a\zeta g + \beta\zeta f - bcg - \beta c\delta, -\alpha\delta E_y \\
&+ (bc + \alpha\zeta)E_z < a\delta e + \alpha c\delta - bef \\
&- \alpha\zeta f, E_y > 0, E_z > 0 \Big\}.
\end{aligned}
$$

Under the above conditions, it follows from (10) that $A_{H5} < 0$, $B_{H5} < 0$, $C_{H5} < 0$, $\Delta_{H5} > 0$, $E_{H5} < 0$, $F_{H5} > 0$, and $G_{H5} < 0$. Therefore, the Jacobian matrix J_{H5} together with the properties satisfy all of the qualitative stability conditions, i.e.,

(i) $A_{H5} < 0$, $E_{H5} < 0$, $G_{H5} < 0$;

(ii) $A_{H5} \neq 0$, $E_{H5} \neq 0$, $G_{H5} \neq 0$;

(iii) $B_{H5}\Delta_{H5} < 0$, $C_{H5}F_{H5} < 0$;

(iv) $B_{H5}0F_{H5} = 0$, $C_{H5}0\Delta_{H5} = 0$, and

(v) $\det(J_{H5}) = (A_{H5}E_{H5}G_{H5} - B_{H5}\Delta_{H5}G_{H5} - C_{H5}E_{H5}F_{H5}) < 0$.

Furthermore we would like to investigate the kind of stability of the equilibrium point $E_H 5$ using Hurwitz stability test. The characteristic equation of the Jacobian matrix J_{H5} evaluated at the equilibrium point $E_H 5$ is

$$
\begin{aligned}
f(r) = r^3 &- (A_{H5} + E_{H5} + G_{H5})r^2 + (A_{H5}E_{H5} \\
&+ A_{H5}G_{H5} - B_{H5}\Delta_{H5} - C_{H5}F_{H5} + E_{H5}G_{H5})r \\
&- A_{H5}E_{H5}G_{H5} + B_{H5}\Delta_{H5}G_{H5} + C_{H5}E_{H5}F_{H5}.
\end{aligned}
$$

Therefore, we have

$$p_0 = \left(- A_{H5}E_{H5}G_{H5} + B_{H5}\Delta_{H5}G_{H5} + C_{H5}E_{H5}F_{H5}\right);$$
$$p_0 > 0,$$
$$p_1 = (A_{H5}E_{H5} + A_{H5}G_{H5} - B_{H5}\Delta_{H5} - C_{H5}F_{H5}$$
$$+ E_{H5}G_{H5}); \quad p_1 > 0,$$
$$p_2 = -\left(A_{H5} + E_{H5} + G_{H5}\right); \quad p_2 > 0, \text{ and}$$
$$p_2 p_1 - p_0 = A_{H5}B_{H5}\Delta_{H5} - A_{H5}{}^2 E_{H5}$$
$$- A_{H5}{}^2 G_{H5} + A_{H5}C_{H5}F_{H5}$$
$$- 2A_{H5}E_{H5}G_{H5} - A_{H5}G_{H5}{}^2$$
$$+ C_{H5}F_{H5}G_{H5} - E_{H5}G_{H5}{}^2$$
$$+ B_{H5}\Delta_{H5}E_{H5} - A_{H5}E_{H5}{}^2 - E_{H5}{}^2 G_{H5};$$
$$p_2 p_1 - p_0 > 0 .$$

$$(11)$$

From (11), following the Hurwitz stability test we conclude that the equilibrium point $E_H 5$ is an attractor trajectory and an asymptotically stable point.

ANALYSIS OF MAXIMUM PROFIT

Now we would like to analyse the requirements of total revenue, total cost, and maximum profit at the equilibrium point $E_H 5$ in order to maximise the profit. In others words, we want to determine the values of efforts E_y and E_z which give maximum profit associated with the stable equilibrium point $E_H 5$.

We assume that the unit price of stocks y and z are p_y and p_z respectively and the total cost is proportional to both of the efforts E_y and E_z with constant coefficients of exploitation c_y and c_z respectively and fixed cost c_f . Thus, the total revenue can be written as a function of E_y and E_z as

$$TR = TR(y_*) + TR(z_*) = p_y E_y y_* + p_z E_z z_*$$

and the total cost written as $TC = c_f + c_y E_y + c_z E_z$. Such assumptions for total revenue function and total cost function have been considered by Clark [12].

The profit function π depends on $y_*, z_*, E_y,$ and E_z . Then we obtain the following expression for the profit function or the nett revenue,

$$\pi = \pi\left(y_*, z_*, E_y, E_z\right)$$
$$= p_y E_y y_* + p_z E_z z_* - c_f - c_y E_y - c_z E_z$$
$$= \left(p_y y_* - c_y\right) E_y + \left(p_z z_* - c_z\right) E_z - c_f .$$

$$(12)$$

From the equilibrium point $E_H 5$, we know that both y_* and z_* depend on E_y and E_z . Thus, we have

$$y_* = \frac{-bcg - \beta c\delta + a\zeta g + \beta\zeta f - (bg + \beta\delta)E_y + \beta\zeta E_z}{beg + a\zeta g + \beta\delta e}$$
$$= y_1 - UE_y + VE_z ,$$

$$(13)$$

where

$$y_1 = \frac{-bcg - \beta c\delta + a\zeta g + \beta\zeta f}{beg + a\zeta g + \beta\delta e} ,$$
$$U = \frac{(bg + \beta\delta)}{beg + a\zeta g + \beta\delta e} ,$$

and

$$V = \frac{\beta\zeta}{beg + a\zeta g + \beta\delta e} ;$$

$$z_* = \frac{-bef - af\zeta + a\delta e + ac\delta - (be + a\zeta)E_z + a\delta E_y}{beg + a\zeta g + \beta\delta e}$$
$$= z_1 - W E_z + X E_y ,$$

$$(14)$$

where

$$z_1 = \frac{-bef - af\zeta + a\delta e + ac\delta}{beg + a\zeta g + \beta\delta e} ,$$
$$W = \frac{(be + a\zeta)}{beg + a\zeta g + \beta\delta e} ,$$

and

$$X = \frac{a\delta}{beg + a\zeta g + \beta\delta e} .$$

Substituting y_* and z_* into profit function (12) will make the profit function dependent only on E_y and E_z . After simplification, the profit function then becomes

$$\pi\left(E_y, E_z\right) = \left(p_y y_1 - c_y\right)E_y - p_y U E_y{}^2$$
$$+ \left(p_y V + p_z X\right)E_y E_z$$
$$+ \left(p_z z_1 - c_z\right)E_z - p_z W E_z{}^2 - c_f$$
$$= y_2\, E_y - p_y U\, E_y{}^2$$
$$+ \left(p_y V + p_z X\right)E_y E_z$$
$$+ z_2\, E_z - p_z W\, E - c_f.$$

$$(15)$$

where $y_2 = \left(p_y y_1 - c_y\right)$ and $z_2 = \left(p_z z_1 - c_z\right)$ are assumed to be positive. Differentiate partially the profit function (15) with respect to E_y and E_z to get the first and second derivatives, we have

$$\frac{\partial \pi}{\partial E_y} = y_2 + \left(p_y V + p_z X\right)E_z - 2p_y U\, E_y,$$

$$\frac{\partial \pi}{\partial E_z} = z_2 + \left(p_y V + p_z X\right)E_y - 2p_z W\, E_z,$$

$$\frac{\partial^2 \pi}{\partial E_y{}^2} = -2p_y U,$$

$$\frac{\partial^2 \pi}{\partial E_z \partial E_y} = \frac{\partial^2 \pi}{\partial E_y \partial E_z} = \left(p_y V + p_z X\right),$$

and

$$\frac{\partial^2 \pi}{\partial E_z{}^2} = -2p_z W.$$

By putting $\dfrac{\partial \pi}{\partial E_y} = \dfrac{\partial \pi}{\partial E_z} = 0$ and solving for E_y

and E_z we get the critical point

$$\left(E_y{}^*, E_z{}^*\right) = \left(\frac{2p_z y_2\, W + z_2\left(p_y V + p_z X\right)}{4p_y p_z\, UW - \left(p_y V + p_z X\right)^2},\right.$$
$$\left.\frac{2p_y z_2\, U + y_2\left(p_y V + p_z X\right)}{4p_y p_z\, UW - \left(p_y V + p_z X\right)^2}\right).$$

We assume $4p_y p_z\, UW - \left(p_y V + p_z X\right)^2 > 0$ to ensure $E_y{}^* > 0$ and $E_z{}^* > 0$. Under this assumption we find that $\dfrac{\partial^2 \pi}{\partial E_y{}^2} < 0$, $\dfrac{\partial^2 \pi}{\partial E_z{}^2} < 0$,

and $\dfrac{\partial^2 \pi}{\partial E_y{}^2}\dfrac{\partial^2 \pi}{\partial E_z{}^2} - \dfrac{\partial^2 \pi}{\partial E_y \partial E_z}\dfrac{\partial^2 \pi}{\partial E_z \partial E_y} > 0$ which are

evaluated at the critical point $\left(E_y{}^*, E_z{}^*\right)$. Thus we verify that the critical point $\left(E_y{}^*, E_z{}^*\right)$ gives the maximum value for the profit function. This

means that if we choose a pair of efforts $\left(E_y{}^*, E_z{}^*\right)$ which belongs to R, the efforts imply the equilibrium point $E_H 5$ is asymptotically stable and also maximise the profit.

Example 2.
Consider the model (8) with parameters $a = 2$, $b = 0.0001$, $\alpha = 0.003$, $c = 0.05$, $e = 0.2$, $f = 0.04$, $g = 0.1$, $\beta = 0.004$, $\zeta = 0.002$, and $\delta = 0.001$. Then we have the equilibrium point $E_H 5 = (x_*, y_*, z_*)$, where

$$x_* = 11778.529 + 88.235E_y + 235.294E_z,$$

$$y_* = 117.535 + 2.353E_z - 4.118E_y,$$

and
$$z_* = 117.3852941 + 0.882352941E_y - 7.647058825E_z.$$

Further we have

$$R = \left\{\left(E_y, E_z\right)\middle|\; 4.118E_y - 2.353E_z\right.$$
$$< 117.535, 7.647E_z - 0.882E_y$$
$$\left.< 117.385, E_y > 0, E_z > 0\right\}$$

Take $p_y = 1$, $p_z = 1$, $c_f = 10$, $c_y = 0.5$, and $c_z = 0.5$. Then we have profit function

$$\pi(E_y, E_z) = 117.03529E_y - 4.1176471E_y{}^2$$
$$+ 3.2352941E_y E_z + 116.88529E_z$$
$$- 7.6470588E_z{}^2 - 10.$$

From the profit function we get the critical point
$$\left(E_y{}^*, E_z{}^*\right) = (18.77404493, 11.61393258)$$
which belongs to R and the maximum profit is
$$\pi(E_y{}^*, E_z{}^*) = 1767.361898.$$

After substituting
$$\left(E_y{}^*, E_z{}^*\right) = (18.77404493, 11.61393258)$$
into the above equilibrium point $E_H 5 = (x_*, y_*, z_*)$, we have
$$E_H 5 = (16167.75281, 67.55730341, 45.13820226)$$
with the Jacobian matrix

$$J_H = \begin{pmatrix} -1.616775281 & -48.50325843 & -64.67101124 \\ 0.1351146068 & -13.51146067 & 0 \\ 0.0451382023 & 0 & -4.51382022 \end{pmatrix},$$

which has eigenvalues $r_1 = -12.9504$ and $r_{2,3} = -3.3459 \pm 1.3225i$. From the eigenvalues we conclude that the equilibrium point $E_H 5$ is asymptotically stable.

CONCLUSIONS

In model (1) the result indicates that when the equilibrium point $E5$ occurs in the positive octane, the equilibrium point $E5$ is asymptotically stable for suitable values of the parameters (Theorem 1). This means that the prey and the two predators can live in coexistence. The three populations tend to the stable equilibrium point $E5$ provided that the populations are initially closed to the stable equilibrium point $E5$.

In the model where the two predators are subjected to harvesting with constant effort, the equilibrium point E_H5 which occurs in the positive octane is asymptotically stable, Theorem 2. This means that the three populations can live in coexistence although the two predators are harvested provided that the efforts of harvesting are controlled. If we take the critical value of harvesting efforts $\left(E_v^{\ast}, E_z^{\ast}\right)$ and choose suitable values of the parameters, we find that the equilibrium point E_H5 which occurs in the positive octane is asymptotically stable and the profit is at the maximum level.

ACKNOWLEDGMENT

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