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MODELLING THE MULTITEAM PREY–PREDATOR DYNAMICS USING THE DELAY DIFFERENTIAL EQUATION

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Abstract: In nature, many species form teams and move in herds from one place to another. This helps them in reducing the risk of predation. Time delay caused by the age structure, maturation period, and feeding time is a major factor in real-time prey–predator dynamics that result in periodic solutions and the bifurcation phenomenon. This study analysed the behaviour of teamed-up prey populations against predation by using a mathematical model. The following variables were considered: the prey population Pr1, the prey population Pr2, and the predator population Pr3. The interior equilibrium point was calculated. A local satiability analysis was performed to ensure a feasible interior equilibrium. The effect of the delay parameter on the dynamics was examined. A Hopf bifurcation was noted when the delay parameter crossed the critical value. Direction analysis was performed using the centre manifold theorem. The graphs of analytical results were plotted using MATLAB.

Keywords: *Prey population, predator population, equilibrium point, delay parameter, direction analysis, hopf bifurcation*

1. Introduction

Numerous applied mathematicians and ecologists have increasingly focused on the predator–prey relationship because of its generality and significance. Several complicated models for two or more interacting species systems have been developed considering the effects of crowding, age structure, time delay, functional response, switching, and other factors (Kesh et al., 2000; Vayenas et al., 2005; Song et al., 2006; Kundu et al., 2018; Kumar et al., 2022).

In the natural world, all species live in the wild. Some live alone, whereas others live in flocks, hives, packs, or herds. For some animals, living with or near other creatures facilitates their survival and ensures that the demands of each individual member are fulfilled through teamwork. Furthermore, building a team and interacting with others is a fundamental tool through which a team member can consistently achieve positive outcomes and readily meet their needs. This study focused on a system in which herds of prey coexist and are attacked by the same type of predator. The problem of multiteam games is relatively new. For certain animals, forming a team more considerably improves their efficiency of food search as a group compared with when this activity is performed alone and reduces the danger of predation.

The interaction of two prey with one predator is often unstable. The probability of the coexistence of all the three species in a given environment is substantially less. Practically, the predator always wins (Poole, 1974). Determining the types of interactions in a multiteam environment can enable researchers to understand the importance of prey teamwork. The finding of such a study would be similar to those reported by Poole on Leslie–Gower computations (Vance, 1978). Examination of the relationship between the prey population and attack rate cannot be beneficial for determining the type of behavioural characteristics exhibited by animals to identify predators (Abrams et al. 1993). By performing an initial assessment of the model's normalised form, a study demonstrated the presence of dynamics in practical predator–prey systems that can be closely represented by fundamental situations (Klebanoff et al., 1994).

A study identified the necessary criteria for all species to survive indefinitely and determined the condition when species becomes extinct in the system (Kesh et al. 2000). By utilising a stochastic logistic differential equation that calculates ecosystem function, a study examined the long-term unexpected behaviour of the at-risk group (Grasman et al., 2001). Another study reported that strong diffusions or interspecific competitions or slower prey intrinsic growth rates and faster predator intrinsic reduction rates are required to obtain a nonconstant solution (Pedersen et al. 2001). To maintain a stable food web, the predator spends its time between preys with different relative densities (Green, 2004). A predator's behaviour towards a certain prey species is affected by the amount of readily digested food.

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These genes and therefore the predator population's lifestyle are regulated by a biosynthetic repression approach (Vayenas et al., 2005). Protection evolution in prey species is improved by survival and decreased density fluctuations based on parameters. The inclusion of a predator's optimal meal choice into the model enhanced cohabitation and reduced overall density variations (Yamauchi et al., 2005). If the impulsive duration surpasses a threshold value, the structure remains typically stable (Song et al., 2006).

Many types of animals prefer to be together in a herd. Because different groups share the same habitat, they may cooperate, compete, or form a predator–prey relationship. New paradigm for predator–prey teams were reported in a previous study (Elettreby, 2009). By applying nonlinear feedback control inputs, three prey–predator populations could be stabilised over time asymptotically. The functional parameter limit is set under which variables converge to limit cycles (El-Gohary et al. 2007).

The system exhibits a stationary distribution that is ergodic under certain conditions. The system's solution remains globally asymptotically stable under certain conditions (Liu et al., 2013). Two preys and one predator comprise a dynamic system modelling multiple teams. During an attack, the individuals of both prey groups would support one another and the pace of predation for both groups would differ (Tripathi et al., 2014). A study proposed and tested a mathematical model of hunting for two competing prey species. The pace of growth and functional responses might be nonlinear functions that are general in nature. The findings suggest the presence of a crucial characteristic governing the system's dynamics that is termed as an intraspecific interference factor (Deka et al., 2016). The criteria for local asymptotic stability were achieved in the lack of climate fluctuation. The authors defined the probabilistic approach by including Gaussian white noise notions into all regular equations (Kundu et al., 2018). The authors evaluated the predator–prey relationship in three species in three dimensions by using an ordinary differential equation. For the calculation, the subjective population sizes of two prey species and one predator creature that share their habitats were considered (Aybar et al. 2018). Prey cooperation benefits both prey populations in many ecosystems. If one prey is harmful and the other is weak for a predator, the predator may continue to follow the weak prey (Mishra et al., 2019). In the presence of only one predator in the system, a fourth-order nonlinear differential equation can be used to represent the system (Zhang et al., 2020). A study used a three-species prey–predator system considering that a predator is layman by nature because it survives on two prey animals (Manna et al., 2020). In this study, we considered two prey populations and two predator populations. When two prey species live in two types of habitats and can defend themselves in groups, only one of the two predators can move between the two types of habitats (Frahan, 2020). Trends of a group and the effectiveness of herbicide are both affected by a predator (Emery et al., 2020).

A one-predator model with temporal delays and a weak Allee effect in the prey's growth function is utilised when two prey populations are engaged in direct competition. Despite its simplicity, the system exhibits a wide range of dynamic behaviour, such as the equilibrium point's biostability (Rihan et al., 2020). A high degree of fear of a prey animal and a higher quality of living for second prey may improve the chances of living of that species (Sahoo et al. 2021). One prey is hazardous, whereas the other is harmless to the predator. The predation processes of both prey teams are independently followed by Monod–Haldane and Holling type II functional responses (Alsakaji et al., 2021). Using stochastic Lyapunov functionals, a study proposed some necessary conditions for extinction and persistence in the mean of the three species (Rihan et al. 2022). To examine delay models in population dynamics, we used the model adopted by Rihan (2021). Few studies have investigated the effect of prey maturation on a predator–prey model through mathematical modelling. Predation of mature prey can be evaluated using delay differential equations (Kumar et al., 2021). Time lag is a crucial factor that should be included in the mathematical model to examine the dynamic behaviour of these types of biological systems (Kumar et al., 2022).

No study has used the delay model for studying the dynamics of prey–predator systems. Thus, this study examined the dynamics of multiteam prey–predator systems by using delay differential equations.

2. Mathematical Model

This study examined the dynamics of a two-prey one-predator delay differential model where both preys support each other to prevent predation. The predator is expected to take time τ during the gestation phase in this scenario. Thus, the model can be represented as follows:

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$$\frac{dP_{r1}(t)}{dt} = a_1 P_{r1}(t)\big(1 - P_{r1}(\text{t})\big) - P_{r1}(t)P_{r3}(t) + P_{r3}(t)P_{r2}(t)P_{r1}(t) \qquad (1)$$

$$\frac{dP_{r2}(t)}{dt} = b_1 P_{r2}(t)\big(1 - P_{r2}(\text{t})\big) - P_{r2}(t)P_{r3}(t) + P_{r3}(t)P_{r2}(t)P_{r1}(t) \qquad (2)$$

$$\frac{dP_{r3}(t)}{dt} = -c_1 P_{r3}^2(t) + d_1 P_{r1}(t-\tau)P_{r3}(t) + e_1 P_{r2}(t-\tau)P_{r3}(t) \qquad (3)$$

where τ > 0 is the lag time necessary for the predator's gestation period; P_{r1} (t) and P_{r2} (t) are the populations of the two teams of preys, respectively; and P_{r3} (t) is the population of predators. All the parameters have positive values, that is, the values of a_1, b_1, c_1, d_1, and e_1 are more than zero. The system (Poole, 1974; Vance, 1978; Abrams et al., 1993) is related to the following starting functions:

$$(P_{r1}(\theta), P_{r2}(\theta), P_{r3}(\theta)) \in C_+ = C\big((-\tau, 0), R_+^3\big), P_{r3}(0),\ P_{r2}(0), P_{r1}(0) > 0$$

The variables and parameters considered in the model by Poole (1974) and Abrams et al. (1993) are listed in Table 1.

Table 1. Description of variables and parameters

Variables/Parameters	Description
P_{r1}	First prey population
P_{r2}	Second prey population
P_{r3}	Predator population
a_1	Natural growth rate of P_{r1}
b_1	Natural growth rate of P_{r2}
c_1	Death rate of the predator population due to mutual competition.
d_1	Rate of predation of P_{r1}
e_1	Rate of predation of P_{r2}
τ	Delay parameter

2.1 Equilibrium Point

The systems (Poole, 1974; Vance, 1978; Abrams et al., 1993) have eight equilibria with specific nonnegativity requirements. In this part, we focus only on the stability and local Hopf bifurcation of the inner equilibrium because the other seven equilibria do not exhibit any effect of delay on the stability of results. To calculate the equilibrium point, equate equation [1] to zero

$$\frac{dP_{r1}}{dt} = 0$$

$$a_1 P_{r1} - a_1 P_{r1}^2 - P_{r1}P_{r3} + P_{r1}P_{r2}P_{r3} = 0$$

$$P_{r1}(a_1 - a_1 P_{r1} - P_{r3} + P_{r2}P_{r3}) = 0$$

$$\text{Either } P_{r1} = 0 \ or\ a_1 - a_1 P_{r1} - P_{r3} + P_{r2}P_{r3} = 0$$

$$\frac{dP_{r2}}{dt} = 0$$

$$b_1 P_{r2} - b_1 P_{r2}^2 - P_{r2}P_{r3} + P_{r1}P_{r2}P_{r3} = 0$$

$$P_{r2}(b_1 - b_1 P_{r2} - P_{r3} + P_{r1}P_{r3} = 0$$

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$$\frac{dP_{r3}}{dt} = 0$$

$$-c_1 P_{r3}^2 + d_1 P_{r1} P_{r3} + e_1 P_{r2} P_{r3} = 0$$

$$P_{r3}(-c_1 P_{r3} + d_1 P_{r1} + e_1 P_{r2}) = 0$$

As $P_{r1} \neq 0, P_{r2} \neq 0, P_{r3} \neq 0$ so we have three equations in P_{r1}, P_{r2}, P_{r3}

$$a_1 - a_1 P_{r1} - P_{r3} + P_{r2} P_{r3} = 0$$

$$a_1(1 - P_{r1}) + (-1 + P_{r2}) P_{r3} = 0 \tag{4}$$

$$b_1 - b_1 P_{r2} - P_{r3} + P_{r1} P_{r3} = 0$$

$$b_1(1 - P_{r2}) + (-1 + P_{r1}) P_{r3} = 0 \tag{5}$$

$$-c_1 P_{r3} + d_1 P_{r1} + e_1 P_{r2} = 0 \tag{6}$$

Multiply (4) by $-(-1 + P_{r1})$ and (5) by $(-1 + P_{r1})$ then add

$$a_1(1 - P_{r1})(-)(-1 + P_{r1}) - (-1 + P_{r1})(-1 + P_{r2}) P_{r3} = 0$$

$$b_1(1 - P_{r1})(-1 + P_{r2}) + (-1 + P_{r1})(-1 + P_{r2}) P_{r3} = 0$$

Add these equations

$$a_1(1 - P_{r1})(1 - P_{r1}) + b_1(1 - P_{r1})(-1 + P_{r2}) = 0$$

$$a_1(1 - P_{r1})^2 - b_1(1 - P_{r2})^2 = 0$$

$$a_1(1 - P_{r1})^2 = b_1(1 - P_{r2})^2$$

$$\frac{a_1}{b_1}(1 - P_{r1})^2 = b_1(1 - P_{r2})^2$$

Taking square root

$$\sqrt{\frac{a_1}{b_1}}(1 - P_{r1}) = (1 - P_{r2})$$

$$P_{r2} = 1 - \sqrt{\frac{a_1}{b_1}}(1 - P_{r1})$$

Put this value in (5)

$$\left[1 - \left\{1 - \sqrt{\frac{a_1}{b_1}}(1 - P_{r1})\right\}\right] b_1 + (-1 + P_{r1}) P_{r3} = 0$$

$$\left[1 - 1 + \sqrt{\frac{a_1}{b_1}}(1 - P_{r1})\right] b_1 + (-1 + P_{r1}) P_{r3} = 0$$

$$b_1 \left[\sqrt{\frac{a_1}{b_1}}(1 - P_{r1})\right] + (-1 + P_{r1}) P_{r3} = 0$$

$$\sqrt{a_1 b_1}(1 - P_{r1}) + (-1 + P_{r1}) P_{r3} = 0$$

$$(-1 + P_{r1}) P_{r3} = -\sqrt{a_1 b_1}(1 - P_{r1})$$

$$(-1 + P_{r1}) P_{r3} = \sqrt{a_1 b_1}(-1 + P_{r1})$$

$$P_{r3} = \sqrt{a_1 b_1}$$

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Let $E^*(P_{r1}^*,\ P_{r2}^*,\ P_{r3}^*)$ denote the interior equilibrium

where

$$P_{r3}^* = \sqrt{a_1 b_1}, \qquad P_{r2}^* = \frac{c_1\sqrt{a_1b_1} - d_1(1-\sqrt{b_1/a_1})}{e_1 + d_1\sqrt{b_1 a_1}}, P_{r1}^* = \frac{b_1c_1 + e_1\left(1-\sqrt{\frac{b_1}{a_1}}\right)}{e_1 + d_1\sqrt{\frac{b_1}{a_1}}}$$

$$c_1\sqrt{a_1b_1} \le d_1 + e, \qquad c_1a_1 + d_1 > d_1\sqrt{b_1/a_1}, \qquad c_1b_1 + e_1 > e_1\sqrt{a_1/b_1}$$

2.2 Stability

Now, we calculate stability of the aforementioned system of equation

$$\frac{dP_{r1}}{dt} = a_1P_{r1} - a_1 P_{r1}^2 - P_{r1}P_{r3} + P_{r1}P_{r2}P_{r3}$$

$$\frac{dP_{r2}}{dt} = b_1P_{r2} - b_1 P_{r2}^2 - P_{r2}P_{r3} + P_{r1}P_{r2}P_{r3}$$

$$\frac{dP_{r3}}{dt} = -c_1 P_{r3}^2 + d_1P_{r1}(t-\tau)P_{r3} + e_1P_{r2}(t-\tau)P_{r3}$$

Differentiation w.r.t. P_{r1}

$$m_1 = a_1 - 2a_1P_{r1} - P_{r3} + P_{r2}P_{r3}, \qquad m_2 = P_{r2}P_{r3}, \qquad m_3 = d_1P_{r3}e^{-\lambda\tau}$$

Differentiation w.r.t. P_{r2}

$$m_4 = P_{r1}P_{r3}, \qquad m_5 = b_1 - 2b_1P_{r2} - P_{r3} + P_{r1}P_{r3}, \qquad m_6 = e_1P_{r3}e^{-\lambda\tau}$$

Differentiation w.r.t. P_{r3}

$$m_7 = -P_{r1} + P_{r1}P_{r2}, \qquad m_8 = -P_{r2} + P_{r1}P_{r2}, \qquad m_9 = -2c_1P_{r3}$$

Let $v_1 = P_{r1} - P_{r1}^*, v_2 = P_{r2}P_{r2}^*$ and $v_3 = P_{r3}P_{r3}^*$then equations [1], [2] and [3] can be expressed in this form

$$\frac{dv_1}{dt} = -a_1P_{r1}^* - P_{r1}^*v_3 + P_{r1}^*P_{r2}^*v_3 + P_{r1}^*P_{r3}^*v_2 - a_1v_1^2 - (1-P_{r2}^*)v_1v_3 + P_{r1}^*v_2v_3 + P_{r3}^*v_1v_2 + v_1v_2v_3 \qquad (7)$$

$$\frac{dv_2}{dt} = -b_1P_{r2}^*v_2 - P_{r2}^*v_3 + P_{r1}^*P_{r2}^*v_3 + P_{r2}^*P_{r3}^*v_1 - b_1v_2^2 - (1-P_{r1}^*)v_2v_3 + P_{r2}^*v_1v_2 + P_{r3}^*v_1v_2 + v_1v_2v_3 \qquad (8)$$

$$\frac{dv_3}{dt} = -c_1P_{r3}^*v_3 - d_1P_{r3}^*v_1(t-\tau) + e_1P_{r3}^*v_2(t-\tau) - c_1v_3^2 - d_1v_1(t-\tau)v_3 + e_1v_2(t-\tau)v_3 \qquad (9)$$

The stability of the equilibrium $E^*(P_{r1}^*,\ P_{r2}^*,\ P_{r3}^*)$ can be examined by investigating the stability of the origin for equations [7], [8], and [9]. Now, we compute the linearised system's characteristics equations [7], [8], and [9] at (0, 0, 0)

$$\begin{vmatrix} \lambda - m_1 & -m_2 & -m_3 \\ -m_4 & \lambda - m_5 & -m_6 \\ -m_7 & -m_8 & \lambda - m_9 \end{vmatrix} = 0$$

$$\begin{vmatrix} \lambda - (a_1 - 2a_1P_{r1} - P_{r3} + P_{r2}P_{r3}) & -P_{r2}P_{r3} & -d_1P_{r3}e^{-\lambda\tau} \\ -P_{r1}P_{r3} & \lambda - (b_1 - 2b_1P_{r2} - P_{r3} + P_{r1}P_{r3}) & -e_1P_{r3}e^{-\lambda\tau} \\ P_{r1} - P_{r1}P_{r2} & P_{r2} - P_{r1}P_{r2} & \lambda + 2c_1P_{r3} \end{vmatrix} = 0$$

After simplification, we obtain the characteristic equation

$$\lambda^3 + X\lambda^2 + Y_1\lambda + e^{-\lambda\tau}(Y_2\lambda + Z_2) = 0 \qquad (10)$$

When τ = 0, equation [10] becomes

$$\lambda^3 + X\lambda^2 + (Y_1 + Y_2)\lambda + Z_2 = 0 \qquad (11)$$

The Routh–Hurwitz criteria implies that with τ = 0, the equilibrium point E^* is locally asymptotically stable if

$$(\boldsymbol{H_1}) X > 0, \qquad (Y_1 + Y_2) > 0, \qquad Z_2 > 0, \qquad X(Y_1 + Y_2) > Z_2 \ hold$$

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Let us suppose that the condition in $(\boldsymbol{H_1})$ is satisfied. Then, equation [11] with $\tau = \tau_j (j = 0, 1, \dots \dots)$ has a simple pair of conjugate purely imaginary roots $\pm i\omega_0$

$$\tau_j = \frac{1}{\omega_0}\left[arccos\frac{\omega_0^2(\omega_0^2 Y_2 + Z_2 X - Y_1 Y_2)}{Y_2^2\omega_0^2 + Z_2^2} + 2j\pi\right]$$

We have the following conditions:

1. If $\tau \epsilon [0, \tau_0)$, all the roots of equation [11] have negative real parts.
2. If $\tau = \tau_0$, the equation [11] has a pair of conjugate purely imaginary roots $\pm i\omega_0$ and real part is negative for all other solutions.

Proof If $\lambda = 0$ is a solution of (11) if Z_2 = 0. This condition contradicts the third requirement stated in $(\boldsymbol{H_1})$, implying that $\lambda = 0$ is not a solution of [11]. Assume that for some $\tau \geq 0, i\omega$ with $\omega > 0$ is a solution of [11].

$$-i\,\omega^3 - X\omega^2 + iY_1\omega + e^{-\omega\tau}(iY_2\omega + Z_2) = 0$$

$$-i\,\omega^3 - X\omega^2 + iY_1\omega + (cos\,\omega\tau - isin\,\omega\tau)\,(iY_2\omega + Z_2) = 0$$

$$i\,(-\omega^3 + Y_1\omega + Y_2\omega cos\,\omega\tau - Z_2 sin\,\omega\tau) + (-X\omega^2 + Z_2 cos\,\omega\tau + Y_2\omega sin\,\omega\tau) = 0$$

Separating real and imaginary parts

$$-\omega^3 + Y_1\omega + Y_2\omega cos\,\omega\tau - Z_2 sin\,\omega\tau = 0 \quad (12)$$

$$-X\omega^2 + Z_2 cos\,\omega\tau + Y_2\omega sin\,\omega\tau = 0 \quad (13)$$

which gives $$\omega^6 + \alpha\omega^4 + \beta\omega^2 + \gamma = 0 \quad (14)$$

where $\alpha = X^2 - 2Y_1, \beta = Y_1^2 - Y_2^2, \gamma = -Z_2^2$

Let $l = \omega^2$, then equation [14] becomes

$$l^3 + \alpha l^2 + \beta l + \gamma = 0 \quad (15)$$

Supposed $l_1, l_2\; and\; l_3$ are the roots of equation [15] and connected by

Sum of the roots $$l_3 + l_2 + l_1 = -\alpha \quad (16)$$

Product of the roots $$l_3\, l_2\, l_1 = -\gamma \quad (17)$$

Thus, equation [17] has either one or three positive real roots.

Depending on the determinant $\Delta_1$of the equation [15]

where $\Delta_1 = \left(\frac{S}{2}\right)^2 + \left(\frac{T}{3}\right)^3$ and $T = \beta - \frac{1}{3}\alpha^2,\ S = \frac{2}{27}\alpha^3 - \frac{1}{3}\alpha\beta + \gamma$

Three situations are possible for the solution of [15]:

a) If $\Delta_1 > 0$, then one real root and a pair of imaginary roots can be obtained for equation [15]. When the real root is positive, it can be written as follows:

$$l_1 = \sqrt[3]{\frac{-S}{2} + \sqrt{\Delta_1}} + \sqrt[3]{\frac{-S}{2} - \sqrt{\Delta_1}} - \frac{1}{3}\alpha$$

b) If $\Delta_1 = 0$, all three real roots and two repeated roots are obtained for equation [15]. If $\alpha > 0$, we obtain only one positive root, $l_1 = 2\sqrt[3]{\frac{-S}{2}} - \frac{1}{3}\alpha. If\ \alpha < 0$, we obtain only one positive root, $l_1 = 2\sqrt[3]{\frac{-S}{2}} - \frac{1}{3}\alpha$ for , $\sqrt[3]{\frac{-S}{2}} > -\frac{1}{3}\alpha$ and three positive real roots for $\frac{\alpha}{6} < \sqrt[3]{\frac{-S}{2}} < -\frac{1}{3}\alpha$, $l_1 = 2\sqrt[3]{\frac{-S}{2}} - \frac{1}{3}\alpha,\ \ l_2 = l_3 = -\sqrt[3]{\frac{-S}{2}} - \frac{1}{3}\alpha$

c) If $\Delta_1 < 0$, we obtain all the three roots are real and distinct, $l_1 = 2\sqrt{\frac{|T|}{3}}\cos\left(\frac{\xi}{3}\right) - \frac{\alpha}{3}$

$$l_2 = \sqrt{\frac{|T|}{3}}cos\left(\frac{\xi}{3} + \frac{2\pi}{3}\right) - \frac{\alpha}{3}, \qquad l_3 = 2\sqrt{\frac{|T|}{3}}\cos\left(\frac{\xi}{3} + \frac{4\pi}{3}\right) - \frac{\alpha}{3}$$

Where $\cos\xi = \left(-\frac{S}{2\sqrt{\left(\frac{|T|}{3}\right)^3}}\right)$, $0 < \xi < \pi$. Moreover, if $\alpha > 0$, only one positive root exists. Otherwise, if $\alpha < 0$, we obtain all three real positive roots or only one positive real root. It is equivalent to max (l_1, l_2, l_3) only when we obtain one positive real root. The number of positive real roots depends on the sign of α. Equation [15] has only one positive real root when $\alpha \geq 0$ is present. Otherwise, we obtained three positive real roots. When $\alpha = X^2 - 2Y_1 > 0$, one positive real root is obtained for [15]. Let the obtained positive real root be denoted by symbol l_0. Then, equation [14] would have only one positive real root $\omega_0 = \sqrt{l_0}$. From equation [13], we have

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$$cos\omega_0\tau = \frac{\omega_0^2(\omega_0^2 Y_2 + Z_2 X - Y_1 Y_2)}{Y_2^2\omega_0^2 + Z_2^2}$$

Express

$$\tau_j = \frac{1}{\omega_0}\left[arccos\frac{\omega_0^2(\omega_0^2 Y_2 + Z_2 X - Y_1 Y_2)}{Y_2^2\omega_0^2 + Z_2^2} + 2j\pi\right] \quad (18)$$

where $j = 0, 1, 2, 3, 4, 5, ...$ $\pm i\omega_0$ obtained a root of equation [10] when $\tau = \tau_j$.
Furthermore, if (H_1) standards are fulfilled, all the roots of equation [10] with $\tau = 0$ have negative real values. We determine the outcomes of lemma1 by summarising the preceding discussion and using the lemma provided. The proof is completed with the following outcomes from theorem lemma 1.

Theorem 2 Assume the condition in (H1) is fulfilled. If $\tau \epsilon [0, \tau_0)$, then the zero solution of equations [7], [8], and [9] is asymptotically stable. Using the classic Hopf bifurcation theorem for retarded functional differential equations, we can obtain the following factors:

Lemma 3. Let $n(l_0) = (3l_0^2 + 2\alpha l_0 + \beta) \neq 0$ and condition in (H_1) are satisfied. For $(j = 0, 1, ...)$, $\lambda(\tau) = \delta(\tau) + i\omega(\tau)$ is denoted as the solution of equation [10] that fulfils the condition $\delta(\tau_j) = 0, \omega(\tau_j) = \omega_0$ were

$$\tau_j = \frac{1}{\omega_0}\left[arccos\frac{\omega_0^2(\omega_0^2 Y_2 + Z_2 X - Y_1 Y_2)}{Y_2^2\omega_0^2 + Z_2^2} + 2j\pi\right]$$

Then, $\pm i\omega_0$ are pair of simple roots. If the transversality condition (H_1) $\delta'(\tau_J) = \left.\frac{Re\,\lambda(\tau)}{d\tau}\right|_{\lambda = i\omega_0} \neq 0$ holds good, we obtain a Hopf bifurcation for [7], (8) and (9) at $v = 0$ and $\tau = \tau_j$.
Proof. Assume $\lambda = \lambda(\tau)$ is a solution of equation [10]. Put $\lambda(\tau)$ in [10] and differentiating with respect to τ on both sides, we get

$$\left[(3\lambda^2 + 2X\lambda + Y_1) + \left((\lambda Y_2 + Z_2)(-\tau) + Y_2\right)e^{-\lambda\tau}\right]\frac{d\lambda}{d\tau} = \lambda(\lambda Y_2 + Z_2)e^{-\lambda\tau}$$

Thus

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(3\lambda^2 + 2X\lambda + Y_1)e^{\lambda\tau}}{\lambda(\lambda Y_2 + Z_2)} + \frac{Y_2}{\lambda(\lambda Y_2 + Z_2)} - \frac{\tau}{\lambda}$$

From [12]–[15], we have

$$-\alpha'(\tau_J) = Re\left[\frac{(3\lambda^2 + 2X\lambda + Y_1)e^{\lambda\tau}}{\lambda(\lambda Y_2 + Z_2)}\right] + Re\left[\frac{Y_2}{\lambda(\lambda Y_2 + Z_2)}\right]$$

$$= \frac{1}{\Omega}[3\omega_0^6 + 2(X^2 - 2Y_1)\omega_0^4 + (Y_1^2 - 2XZ_2 - Y_2^2)\omega_0^2]$$

$$= \frac{1}{\Omega}(3\omega_0^6 + 2\alpha\omega_0^4 + \beta\omega_0^2)$$

$$= \frac{\omega_0^2}{\Omega}(3\omega_0^4 + 2\alpha\omega_0^2 + \beta)$$

$$= \frac{\omega_0^2}{\Omega}(3l_0^2 + 2\alpha l_0 + \beta)$$

$$= \frac{\omega_0^2}{\Omega}n(l_0)$$

where $\Omega = Y_2^2\omega_0^2 + Z_2^2$

$n(l_0) = 3l_0^2 + 2\alpha l_0 + \beta$. Observed, when $\Omega > 0$ and $\omega_0 > 0$

We find that

Sign $[\delta'(\tau_J)] = sign[n(l_0)]$ proves the theorem.

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3. Direction Analysis and Stability of the Hopf Bifurcation Solution

In the previous section, we observed that a set of solutions can be obtained as bifurcates from the favourable steady state E* at a crucial level of τ. The direction, stability, and period of these bifurcating periodic solutions should be determined. We build precise equations defining the properties of the Hopf bifurcation at the critical value by using normal form theory and the centre manifold theorem at the critical point τ_j in this section.

Normalizing delay value τ by the time scaling $t \longrightarrow \frac{t}{\tau}$ system [7], [8], and [9] is transformed into

$$\frac{dv_1}{dt} = -a_1P_{r1}^*v_1 - P_{r1}^*v_3 + P_{r1}^*P_{r2}^*v_3 + P_{r1}^*P_{r3}^*v_2 - av_1^2 - (1-P_{r2}^*)v_1v_3 + P_{r1}^*v_2v_3 + P_{r3}^*v_1v_2 + v_1v_2v_3 \quad (19)$$

$$\frac{dv_2}{dt} = -b_1P_{r2}^*v_2 - P_{r2}^*v_3 + P_{r1}^*P_{r2}^*v_3 + P_{r2}^*P_{r3}^*v_1 - bv_2^2 - (1-P_{r1}^*)v_2v_3 + P_{r2}^*v_1v_3 + P_{r3}^*v_1v_2 + v_1v_2v_3 \quad (20)$$

$$\frac{dv_3}{dt} = -c_1P_{r3}^*v_3 + d_1P_{r3}^*v_1(t-1) + e_1P_{r3}^*v_2(t-1) - c_1v_3^2 + d_1v_1(t-1)v_3 + e_1v_2(t-1)v_3 \quad (21)$$

Take phase plane $C = C((-1,0), R_+^3)$. WLOG, denote the critical value τ_j by τ_0. If $\tau = \tau_0 + \sigma$, then $\sigma = 0$ is a value for Hopf bifurcation for equations [19]–[21]. For the simplicity of notations, we rewrite [19]–[21] in this form

$$v'(t) = L_\sigma(v_t) + G(\sigma, v_t) \quad (22)$$

where $v(t) = v_1(t), v_2(t), v_3(t) \epsilon R^3, v_t(\theta) = v(t+\theta)$ and

$$L_\sigma\varphi = (\tau_0+\sigma)\begin{bmatrix} -a_1P_{r1}^* & P_{r1}^*P_{r3}^* & -P_{r1}^* + P_{r1}^*P_{r2}^* \\ P_{r2}^*P_{r3}^* & -b_1P_{r2}^* & -P_{r2}^* + P_{r1}^*P_{r2}^* \\ 0 & 0 & -c_1P_{r3}^* \end{bmatrix}\begin{bmatrix} \varphi_1(0) \\ \varphi_2(0) \\ \varphi_3(0) \end{bmatrix} +$$

$$(\tau_0+\sigma)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d_1P_{r3}^* & e_1P_{r3}^* & 0 \end{bmatrix}\begin{bmatrix} \varphi_1(-1) \\ \varphi_2(-1) \\ \varphi_3(-1) \end{bmatrix}$$ and

$$G(\sigma,\varphi) = (\tau_0+\sigma)\begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}$$ respectively, were

$$G_1 = -a_1\varphi_1^2(0) - (1-P_{r2}^*)\varphi_1(0)\varphi_3(0) + P_{r1}^*\varphi_2(0)\varphi_3(0) + P_{r3}^*\varphi_2(0)\varphi_1(0) + \varphi_3(0)\varphi_2(0)\varphi_1(0),$$
$$G_2 = -b_1\varphi_1^2(0) - (1-P_{r1}^*)\varphi_3(0)\varphi_2(0) + P_{r2}^*\varphi_3(0)\varphi_1(0) + P_{r3}^*\varphi_2(0)\varphi_1(0) + \varphi_3(0)\varphi_2(0)\varphi_1(0),$$
$$G_3 = -c_1\varphi_3^2(0) + d_1\varphi_1(-1)\varphi_3(0) + c_1\varphi_2(-1)\varphi_3(0),$$
$$\varphi(0) = (\varphi_1(\theta), \varphi_2(\theta), \varphi_3(\theta))^T \epsilon\, C(C-1,0), R).$$

Using the Riesz representation theorem, we can find a function $\sphericalangle(\theta,\sigma)$ of bounced variation for $\theta \epsilon [-1,0)$ as

$$L_\sigma\varphi = \int_{-1}^{0} d\sphericalangle(\theta,0)\,\varphi(0) for\ \varphi \epsilon C.$$

We choose

$$\sphericalangle(\theta,\sigma) = (\tau_0+\sigma)\begin{bmatrix} -a_1P_{r1}^* & P_{r1}^*P_{r3}^* & -P_{r1}^* + P_{r2}^*P_{r1}^* \\ P_{r2}^*P_{r3}^* & -b_1P_{r2}^* & -P_{r2}^* + P_{r2}^*P_{r1}^* \\ 0 & 0 & -c_1P_{r3}^* \end{bmatrix}\chi(\theta) +$$

$$(\tau_0+\sigma)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d_1P_{r3}^* & e_1P_{r3}^* & 0 \end{bmatrix}\chi(\theta+1)$$

where χ is the Delta Dirac function for $\varphi \epsilon C([-1,0], R_+^3)$,

Let us define a function

$$A(\sigma)\varphi = \begin{cases} d\varphi(0) & \theta \epsilon [-1,0) \\ \int_{-1}^{0} d \sphericalangle (\theta,\varphi) & \theta = 0 \end{cases}$$

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$$H(\sigma)\varphi = \begin{cases} 0 & \theta \epsilon[-1,0) \\ G(\sigma,\varphi) & \theta = 0 \end{cases}$$

Then, the system (22) is equivalent to

$$v'_t = X(\sigma)v_t + H(\sigma)v_t \tag{23}$$

For $\varepsilon \epsilon C'([-1,0], R_+^3)$, define

$$X^*\epsilon(s) = \begin{cases} \frac{-d\epsilon(s)}{d\theta} & s \epsilon [-1,0, \\ \int_{-1}^{0} d \measuredangle^T (-1,0)\varepsilon(-t) & s = 0 \end{cases}$$

The bilinear inner product is as follows:

$$< \varepsilon(s), \varphi(\theta) > = \in (0)\,\varphi(0) - \int_{-1}^{0}\int_{v=0}^{0} \epsilon(v-\theta) d \measuredangle (\theta)\varphi(v)dv \tag{24}$$

X^* *and* $X(0)$are adjoint operators; thus, $i\omega_0$ are the eigen values of $X(0)$. They are the eigen values of X^*. Suppose that $\beta(\theta) = \beta(0)e^{i\omega_0\theta}$is an eigen vector of $X(0)$ corresponding to the eigenvalue $i\omega_{0.}$ Then, $X(0) = i\omega_0\beta(\theta)$. When $\theta = 0$, we obtain

$$\left[i\omega_0 I - \int_{-1}^{0} d \measuredangle (\theta)e^{i\omega_0\theta}\right]\beta(0) = 0$$

which yields $\beta(0) - (1, x_1 y_1)^T$,
where

$$x_1 = \frac{(P_{r1}^* - P_{r2}^*P_{r1}^*)P_{r3}^*P_{r2}^* + (P_{r2}^* - P_{r2}^*P_{r1}^*)(i\omega_0 + a_1P_{r1}^*)}{P_{r1}^*P_{r3}^*(P_{r2}^* - P_{r2}^*P_{r1}^*) - (P_{r1}^* - P_{r2}^*P_{r1}^*)(i\omega_0 + P_{r2}^*b_1)}$$

$$y_1 = \frac{P_{r2}^*P_{r3}^{*2}P_{r1}^* - (i\omega_0 + a_1P_{r1}^*)(i\omega_0 + b_1P_{r2}^*)}{P_{r1}^*P_{r3}^*(P_{r2}^* - P_{r2}^*P_{r1}^*) - (P_{r1}^* - P_{r2}^*P_{r1}^*)(i\omega_0 + P_{r2}^*b_1)}$$

Similarly, it can be verified that $\beta^*(s) = D(1, x_2 y_2)e^{i\omega_0\tau_0 s}$is the eigen vector of X^*corresponding to $-i\omega_{0,}$
where

$$x_2 = \frac{P_{r2}^*P_{r3}^*(P_{r1}^* - P_{r2}^*P_{r1}^*) + (P_{r2}^* - P_{r2}^*P_{r1}^*)(a_1P_{r1}^* - i\omega_0)}{P_{r1}^*P_{r3}^*(P_{r2}^* - P_{r1}^*P_{r2}^*) - (P_{r1}^* - P_{r1}^*P_{r2}^*)(b_1P_{r2}^* - i\omega_0)}$$

$$y_2 = \frac{P_{r1}^*P_{r2}^*P_{r3}^{*2} - (a_1P_{r1}^* - i\omega_0)(b_1P_{r2}^* - i\omega_0)}{P_{r1}^*P_{r3}^*(P_{r2}^* - P_{r2}^*P_{r1}^*) - (P_{r1}^* - P_{r2}^*P_{r1}^*)(b_1P_{r2}^* - i\omega_0)}$$

To assume $< \beta^*(s), \beta(\theta) \geq 1$, we have to calculate the value of D. From [24], we obtain $< \beta^*(s), \beta(\theta) >$

$$= \bar{D}(1, \bar{x}_2, \bar{y}_2)(1, x_1, y_1)^T - \int_{-1}^{0}\int_{v=0}^{\theta} \bar{D}(1, \bar{x}_2, \bar{y}_2)\, e^{-i\omega_0\tau_0(v-\theta)} d \measuredangle (\theta)(1, x_1, y_1)^T e^{-\omega_0\tau_0} dv$$

$$= \bar{D}\{1 + x_1\bar{x}_2 + y_1\bar{y}_2 - \int_{-1}^{0} (1, \bar{x}_2, \bar{y}_2)\, \theta e^{i\omega_0\tau_0\theta} d \measuredangle (\theta)(1, x_1, y_1)^T\}$$

$$= \bar{D}\{1 + x_1\bar{x}_2 + y_1\bar{y}_2 - \tau_0\, \bar{y}_2 P_{r3}^*(d_1x_1 + e_1y_2)e^{i\omega_0\tau_0}\}$$

Hence, we can choose

$$\bar{D} = \frac{1}{1 + x_1\bar{x}_2 + y_1\bar{y}_2 + \tau_0\bar{y}_2P_{r3}^*(d_1\sigma_{1+}e_1y)e^{i\omega_0\tau_0}}$$

Such that

$$< \beta^*(s), \beta(\theta) \geq 1, < \beta^*(s), \beta(\theta) = 0$$

Continue the coordinates defining the vector by following the algorithm and using the same notations as their manifold c_0 at $\sigma = 0$. Let v_t be a solution of equation [23] with $\sigma = 0$. Define

$$m(t) = < \beta^*(s), v_t(\theta) > \tag{25}$$

$$V(t,\theta) = v_t(\theta) - 2Re\, m(t)\beta(\theta) \tag{26}$$

According to manifold, we obtain centre C_0. Accordingly,

$$V(t,\theta) = V(m(t)\, \bar{m}(t), \theta),$$

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where

$$V(m,\bar{m},\theta) = V_{20}(\theta)\frac{m^2}{2} + V_{11}(\theta)m\bar{m} + V_{02}(\theta)\frac{\bar{m}^2}{2} + \cdots$$

m and $\bar{m}$ are local values for the manifold centre C_0 in the direction of β^* and $\bar{\beta}^*$. When V is real, v_t is real. We assume only the real solution. For solution $v_t \in C_0$ of [23], since $\sigma = 0$,

$$m'(t) = i\omega_0\tau_0 m + < \bar{\beta}^*(\theta), G(0, V(m,\bar{m},\theta) + 2Re\{m(t)\beta(\theta)\}) >$$
$$= i\omega_0\tau_0 m + \bar{\beta}^*(0)G(0, V(m,\bar{m},0) + 2Re\{m(t)\beta(\theta)\})$$
$$= i\omega_0\tau_0 m + \bar{\beta}^*(0)\, G_0(m,\bar{m}) \quad (27)$$

where $s(m,\bar{m}) = \bar{\beta}^*(0)\ \ G_0(m,\bar{m})$

$$= s_{20}(\theta)\frac{m^2}{2} + s_{11}(\theta)m\bar{m} + s_{02}(\theta)\frac{\bar{m}^2}{2} + s_{21}\frac{m^2\bar{m}}{2} + \cdots \quad (28)$$

Noticing

$$v_t(\theta) = (v_{it}, v_{2t}, v_{3t}) = V(t,\theta) + m\beta(\theta) + \bar{m}\bar{\beta}(\theta)$$

and $\beta(0) = (1, x_1, y_1)^T e^{i\omega_0\tau_0\theta}$, we have

$$v_{1t}(0) = m + \bar{m} + V_{20}^{(1)}\frac{m^2}{2} + V_{11}^{(1)}(0)m\bar{m} + V_{02}^{(1)}(0)\frac{\bar{m}^2}{2} + \cdots,$$

$$v_{2t}(0) = x_1 m + \bar{x}_1\bar{m} + V_{20}^{(2)}\frac{m^2}{2} + V_{11}^{(2)}(0)m\bar{m} + V_{02}^{(2)}(0)\frac{\bar{m}^2}{2} + \cdots,$$

$$v_{3t}(0) = y_1 m + \bar{y}_1\bar{m} + V_{20}^{(3)}(0)\frac{m^2}{2} + V_{11}^{(3)}(0)m\bar{m} + V_{02}^{(3)}(0)\frac{\bar{m}^2}{2} + \cdots,$$

$$v_{1t}(-1) = me^{-i\omega_0\tau_0} + \bar{m}e^{i\omega_0\tau_0} + V_{20}^{(1)}(-1)\frac{m^2}{2} + V_{11}^{(1)}(-1)m\bar{m} + V_{02}^{(1)}(-1)\frac{\bar{m}^2}{2} + \cdots,$$

$$v_{2t}(-1) = x_1 e^{-i\omega_0\tau_0} + \bar{x}e^{i\omega_0\tau_0}\bar{m} + V_{20}^{(2)}(-1)\frac{m^2}{2} + V_{11}^{(2)}(-1)m\bar{m} + V_{02}^{(2)}(-1)\frac{\bar{m}^2}{2} + \cdots,$$

Comparing coefficients with (28), we have

$s_{20} = -2\tau_0\bar{D}[a_1 + (1 - P_{r2}^*)y_1 - x_1(P_{r1}^*y_1 + P_{r3}^*) + \bar{x}_2(b_1x_1^2 + (1 - P_{r1}^*)x_1y_1) - x_1P_{r3}^* - y_1P_{r2}^* + \bar{y}_2y_1(c_1y_1 - d_1e^{-i\omega_0\tau_0} - e_1x_1e^{-i\omega_0\tau_0})],$

$s_{11} = -2\tau_0\bar{D}[a_1 + (1 - P_{r2}^*)Re\{y_1\} - P_{r1}^*Re\{\bar{y}_1x_1\} - P_{r3}^*Re\{x_1\} + \bar{x}_2(x_1\bar{x}_1b_1 + (1 - P_{r1}^*)Re\{x_1\bar{y}_1\} - P_{r2}^*Re\{\bar{y}_1\} - P_{r3}^*\{x_1\} + \bar{y}_2(c_1y_1\bar{y}_1 - d_1Re\{y_1e^{i\omega_0\tau_0}\} - e_1Re\{y_1\bar{x}_1e^{i\omega_0\tau_0}\})],$

$s_{02} = -2\tau_0\bar{D}[a_1 + (1 - P_{r2}^*)\bar{y}_1 - \bar{x}_1(P_{r1}^*\bar{y}_1 + P_{r3}^*) + \bar{x}_2(b_1\bar{x}_1^2 + (1 - P_{r1}^*)\bar{x}_1\bar{y}_1) - \bar{x}_1P_{r3}^* - \bar{y}_1P_{r2}^* + \bar{y}_2\bar{y}_1(c_1\bar{y}_1 - d_1e^{-i\omega_0\tau_0})]$

$s_{21} = -2\tau_0\bar{D}[a_1\left(V_{20}^{(1)}(0) + 2V_{11}^{(1)}(0)\right) + (1 - P_{r2}^*)(\frac{1}{2}V_{20}^{(1)}(0)\bar{y}_1 + V_{11}^{(1)}(0)y_1 + \frac{1}{2}V_{20}^{(3)}(0) + V_{11}^{(3)}(0) - (2Re\{x_1\bar{y}_1\} + x_1y_1) - P_{r1}^*\left(\frac{1}{2}W_{20}^{(2)}(0)\bar{y}_1 + \frac{1}{2}V_{20}^{(3)}(0)\bar{x}_1 + V_{11}^{(2)}(0)y_1 + V_{11}^{(3)}(0)x_1\right) - P_{r3}^*\left(\frac{1}{2}V_{20}^{(2)}(0) + \frac{1}{2}V_{20}^{(1)}(0)\bar{x}_1 + V_{11}^{(2)}(0) + V_{11}^{(1)}(0)x_1\right) + \bar{x}_2\left(b_1V_{20}^{(2)}(0)\bar{x}_1 + 2V_{11}^{(2)}(0)x_1\right) + (1 - P_{r1}^*)(\frac{1}{2}V_{20}^{(2)}(0)\bar{y}_1 + V_{11}^{(3)}(0)x_1 - (2Re\{x_1\bar{y}_1\} + x_1y_1) - P_{r2}^*\left(\frac{1}{2}V_{20}^{(1)}(0)\bar{y}_1 + \frac{1}{2}V_{20}^{(3)}(0) + V_{11}^{(1)}(0)y_1 + V_{11}^{(1)}(0)y_1 + V_{11}^{(3)}(0) - P_{r3}^*\left(\frac{1}{2}V_{20}^{2}(0) + \frac{1}{2}V_{20}^{(1)}(0)\bar{x}_1 + V_{11}^{(2)}(0) + V_{11}^{(1)}(0)x_1\right)\right) + \bar{y}_2(c_1\left(V_{20}^{(3)}(0)\bar{y}_1 + 2V_{11}^{(3)}(0)y_1\right) - d_1\left(\frac{1}{2}V_{20}^{(1)}(-1)\bar{y}_1 + V_{11}^{(1)}(-1)y_1 + \frac{1}{2}V_{20}^{(3)}(0)e^{i\omega_0\tau_0} + V_{11}^{(3)}(0)e^{-i\omega_0\tau_0}\right) - e_1(\frac{1}{2}V_{20}^{(1)}(-1)\bar{y}_1 + V_{11}^{(2)}(-1)y_1 + \frac{1}{2}V_{20}^{(3)}(0)\bar{x}_1e^{i\omega_0\tau_0} + V_{11}^{(3)}x_1e^{-i\omega_0\tau_0}))]$

Because of the presence of $V_{20}(\theta)$ and $V_{11}(\theta)$ in s_{21}, we need to further compute them. From (23) and (26), we have

$$V' = v_t' - m'\beta - \bar{m}'\beta$$
$$= \begin{cases} XV - 2Re\{\bar{\beta}^*(0)G_0\beta(\theta)\}, & \theta \in [-1, 0), \\ XV - 2Re\{\bar{\beta}^*(0)G_0\beta(0)\} + G_0 & \theta = 0 \end{cases}$$
$$\triangleq XV + N(m,\bar{m},\theta), \quad (29)$$

where

$$N(m,\overline{m},\theta) = N_{20}(\theta)\frac{m^2}{2} + N_{11}(\theta)m\overline{m} + N_{02}(\theta)\frac{\overline{m}^2}{2} + N_{21}\frac{m^2\overline{m}}{2} + .., \quad (29)$$

On the other hand, on C_0near the origin

$$V' = V_m m' + V_{\overline{m}}\overline{m}'$$

Expanding the aforementioned series and comparing the coefficient, we obtain

$$[X - 2i\omega_0 I]V_{20}(\theta) = -N_{20}(\theta) \quad (30)$$

$$XV_{11}(\theta) = -N_{11}(\theta) \quad (31)$$

From (27), we know that for $\theta \epsilon [-1, 0)$,

$$N(m,\overline{m},\theta) = -\beta^{\bar{y}}(0)\bar{G}_0\beta(\theta) - \beta^*(0)\bar{G}_0\bar{\beta}(\theta) = -s\beta(\theta) - \bar{s}\bar{\beta}(\theta).$$

Comparing the coefficient with (30), we obtain $\theta \epsilon [-1, 0]$ that

$$N_{20}(\theta) = -s_{20}\beta(\theta) - \bar{s}_{02}\bar{\beta}(\theta)$$

$$N_{11}(\theta) = -s_{11}\beta(\theta) - \bar{s}_{11}\bar{\beta}(\theta)$$

From (29), (30), and (31) and the definition of X, we obtain

$$V_{20}(\theta) = 2i\omega_0\tau_0 V_{20}(\theta) + s_{20}\beta(\theta) + \bar{s}_{02}\bar{\beta}(\theta)$$

Solving for $V_{20}(\theta)$, we obtain

$$V_{20}(\theta) = \frac{is_{20}}{\omega_0\tau_0}\beta(0)e^{i\omega_0\tau_{0\theta}} + \frac{i\bar{s}_{02}\bar{\beta}(0)}{3\omega_0\tau_0}e^{-i\omega_0\tau_0\theta} + P_1 e^{2i\omega_0\tau_0\theta}$$

Similarly,

$$V_{11}(\theta) = \frac{-is_{11}}{\omega_0\tau_0}\beta(0)e^{i\omega_0\tau_{0\theta}} + \frac{i\bar{s}_{11}\bar{\beta}(0)}{\omega_0\tau_0}e^{-i\omega_0\tau_0\theta} + P_2$$

where $P_1$and P_z are both three-dimensional vectors and can be calculated by setting $\theta = 0$ in N. Accordingly, we obtain

$$N(m,\overline{m},\theta) = -2Re\{\bar{\beta}^*(0)G_0\beta(0)\} + G_0$$

when

$$N_{20}(\theta) = -s_{20}\beta(\theta) - \bar{s}_{02}\bar{\beta}(\theta) + G_{m^2}$$

$$N_{11}(\theta) = -s_{11}\beta(\theta) - \bar{s}_{11}\bar{\beta}(\theta) + F_{m\overline{m}}$$

where

$$G_0 = G_{m^2}\frac{m^2}{2} + G_{m\overline{m}}m\overline{m} + G_{\overline{m}^2}\frac{\overline{m}^2}{2} + \cdots$$

Combining the definition of X, we obtain

$$\int_{-1}^{0} d \sphericalangle (\theta)V_{20}(\theta) = 2i\omega_0\tau_0 V_{20}(0) + s_{20}\beta(0) + \bar{s}_{02}\bar{\beta}(0) - G_{m^2}$$

and

$$\int_{-1}^{0} d \sphericalangle (\theta)V_{11}(\theta) = s_{11}\beta(0) - \bar{s}_{11}\bar{\beta}(0) - G_{m\overline{m}}$$

Notice that

$$(i\omega_0\tau_0 I - \int_{-1}^{0} e^{i\omega_0\tau_0\theta} d \sphericalangle (\theta))\beta(0) = 0$$

and

$$-i\omega_0\tau_0 I - \int_{-1}^{0} e^{-i\omega_0\tau_0\theta}\ d \sphericalangle (\theta))\bar{\beta}(0) = 0$$

We have

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$$(2i\omega_0\tau_0 I - \int_{-1}^{0} e^{2i\omega_0\tau_0\theta} d \nless (\theta))P_1 = G_{m^2}$$

Similarly, we have

$$-\left(\int_{-1}^{0} d \nless (\theta)\right) P_2 = G_{m\bar{m}}$$

Hence, we obtain

$$\begin{bmatrix} 2i\omega_0 + a_1 P_{r1}^* & -P_{r1}^* P_{r3}^* & P_{r1}^* - P_{r2}^* P_{r1}^* \\ -P_{r3} P_{r2}^* & 2i\omega_0 + b_1 P_{r2}^* & P_{r2}^* - P_{r2}^* P_{r1}^* \\ -d_1 P_{r3}^* e^{-2i\omega_0\tau_0} & -e_1 P_{r3}^* e^{-2i\omega_0\tau_0} & 2i\omega_0 + c_1 P_{r3}^* \end{bmatrix} P_1 = -2\begin{bmatrix} a_1 + (1 - P_{r2}^*)y_1 - x_1(P_{r1}^* y_1 + P_{r3}^*) \\ b_1 x_1^2 + (1 - P_{r1}^*) x_1 y_1) - x_1 P_{r3}^* - y_1 P_{r2}^* \\ y_1(c_1 y_1 - d_1 e^{-i\omega_0\tau_0} - e_1 x_1 e^{-i\omega_0\tau_0}) \end{bmatrix}$$

and

$$\begin{bmatrix} a_1 P_{r1}^* & -P_{r1}^* P_{r3}^* & -P_{r2}^* P_{r1}^* + P_{r1}^* \\ -P_{r2}^* P_{r3} & b_1 P_{r2}^* & -P_{r2}^* P_{r1}^* + P_{r2}^* \\ -d_1 P_{r3}^* & -e_1 P_{r3}^* & c_1 P_{r3}^* \end{bmatrix} P_2 = -2\begin{bmatrix} a_1 + (1 - P_{r2}^*) Re\{y_1\} - P_{r1}^* Re\{\bar{y}_1 x_1\} - P_{r3}^* Re\{x_1\} \\ x_1 \bar{x}_1 b_1 + (1 - P_{r1}^*) Re\{x_1 \bar{y}_1\} - P_{r2}^* Re\{\bar{y}_1\} - P_{r3}^* Re\{x_1\}) \\ c_1 y_1 \bar{y}_1 - d_1 Re\{y_1\} - e_1 Re\{y_1 \bar{x}_1\} e^{i\omega_0\tau_0}) \end{bmatrix}$$

Then, s_{21}can be denoted by the variables.

We determined that s_{ij}can be calculated using the variables. Thus, we computed these quantities as follows:

$$Z_2(0) = \frac{i}{2\omega_0\tau_0}\left(s_{11}s_{20} - 2|s_{11}|^2 - \frac{|s_{02}|^2}{3}\right) + \frac{s_{21}}{2} \quad (32)$$

$$\sigma_2 = -\frac{Re\{Z_2(0)\}}{Re\{\lambda'(\tau_0)\}} \quad (33)$$

$$\beta_2 = 2Re\{Z_2(0)\}$$

$$T_2 = -\frac{I_m\{Z_2(0)\} + \sigma_2 I_m\{\lambda'(Z_0)\}}{\tau_0\omega_0} \quad (34)$$

Theorem. σ_2 calculates the direction of the Hopf bifurcation: if σ_2 <0($\sigma_2 > 0$), we obtain the supercritical Hopf bifurcation. When $\tau > \tau_0$ ($\tau < \tau_0$), we observed the bifurcating period solutions. P_2 indicates that the bifurcating periodic solution is stable. If $\beta_2 < 0$ ($\beta_2 > 0$), we observe that bifurcating periodic solutions are arbitrary and asymptotically stable (unstable). The bifurcating periodic solution is determined by T_2. When T_2> 0 (T_2 < 0), the period increases (decreases), respectively.

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4. Numerical Example

In this part, we used MATLAB to perform a numerical simulation of the system (Poole, 1974; Abrams et al., 1993). We use these parametric values:

Set 1

$$(a_1 = 1.2;\ b_1 = 1.4;\ c_1 = 1;\ d_1 = 1;\ e_1 = 2)$$

We can observe the positive interior equilibrium point when the initial value is 0.2, 0.4, or 0.6.

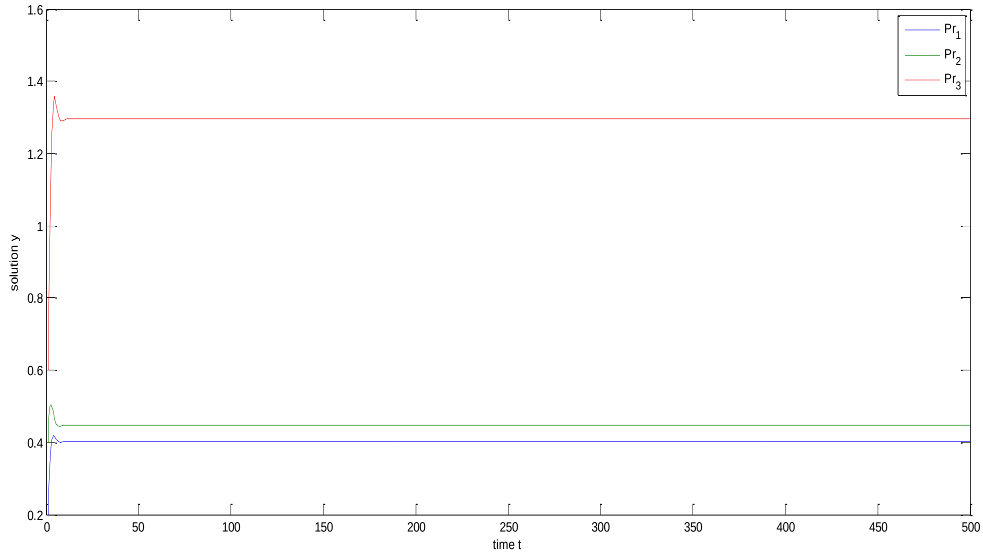

Figure 1. In the absence of delay, the system is stable

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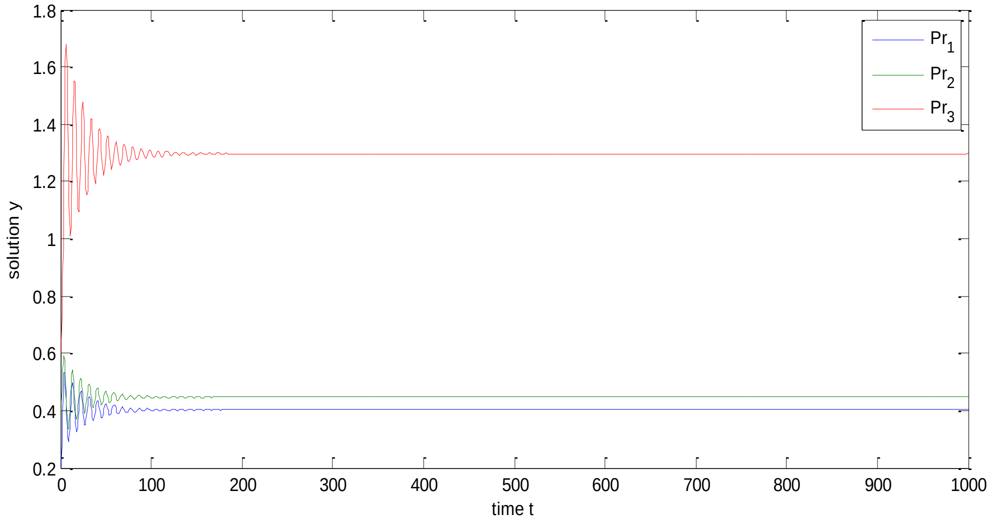

Figure 2. Asymptotically stable when $\tau = 1.5 < \tau_0 = 1.7387$

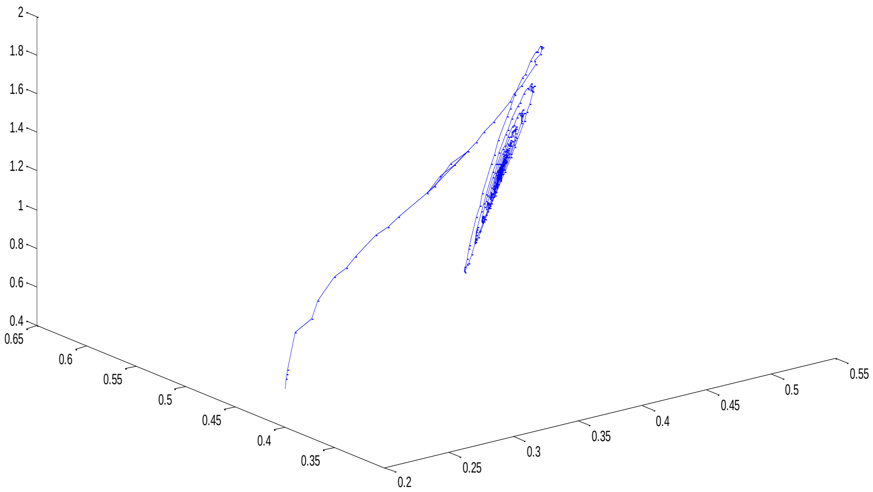

Figure 2.1 Phase plane graph for asymptotically stable when $\tau = 1.5 < \tau_0 = 1.7387$

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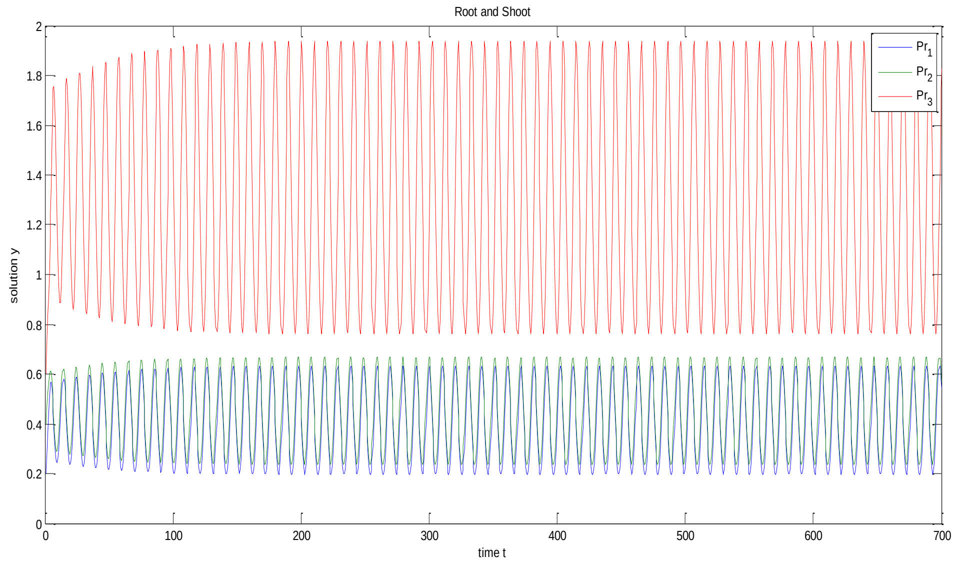

Figure 3. Hopf Bifurcation when $\tau = 1.85 > \tau_0 = 1.7387$

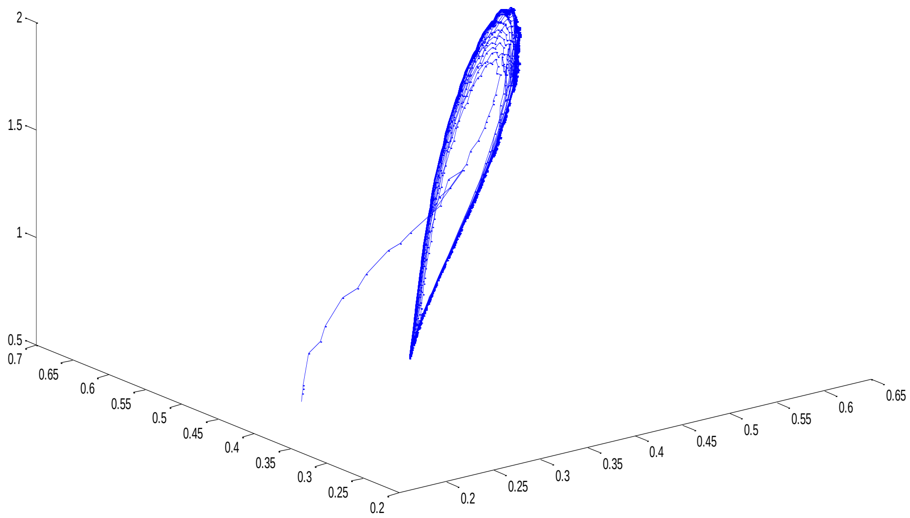

Figure 3.1 Phase plane graph for Hopf Bifurcation when $\tau = 1.85 > \tau_0 = 1.7387$

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Set 2 (a_1 = 1; b_1 = 1.44; c_1 = 1; d_1= 1; e_1 =1.2)
The positive interior equilibrium point is obtained when the initial value is 0.2, 0.4, and 0.6.

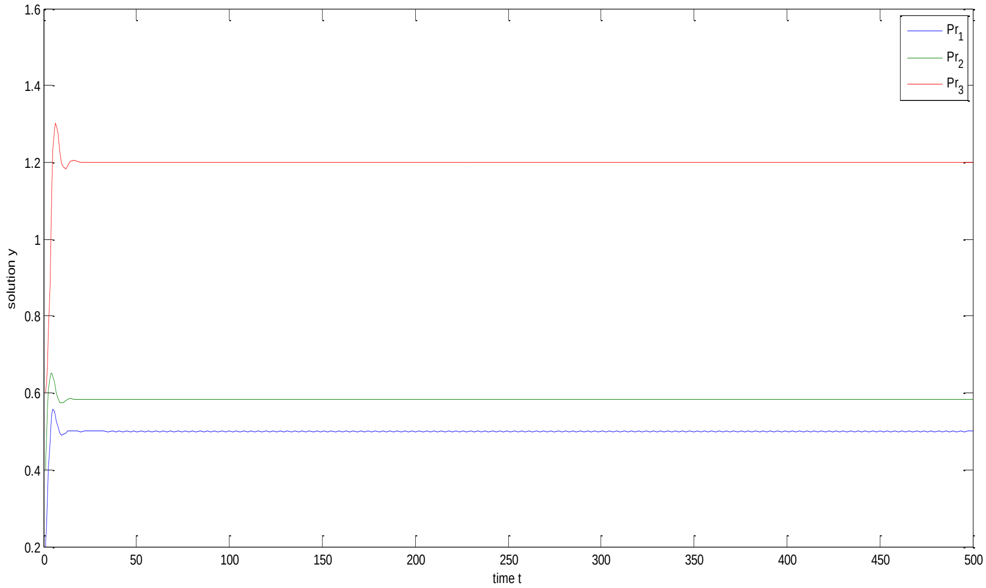

Figure 4. In the absence of delay, the system is stable

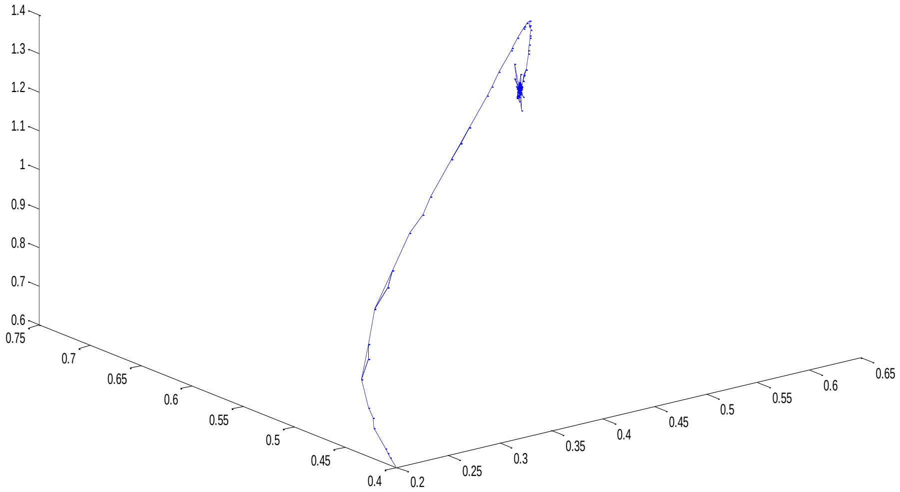

Figure 4.1 Phase plane graph in the absence of delay in the system

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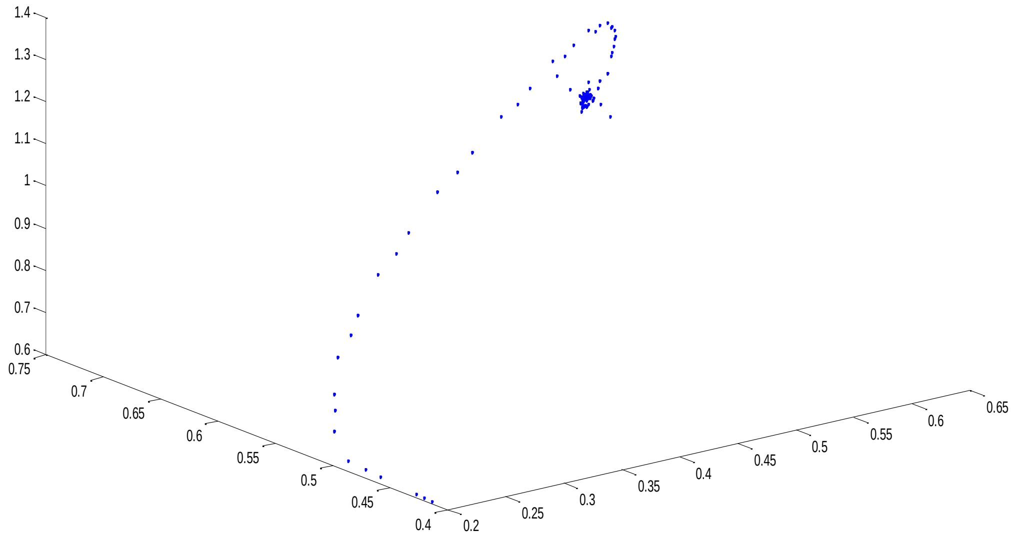

Figure 4.2 Phase plane graph in the absence of delay in the system

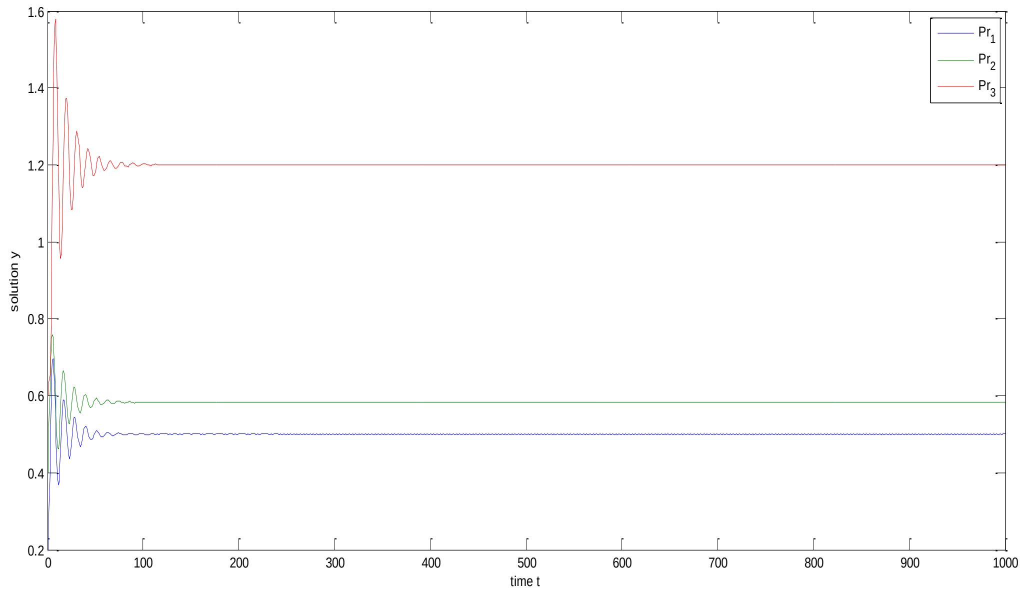

Figure 5. Asymptotically stable when $\tau < \tau_0 = 1.7387$

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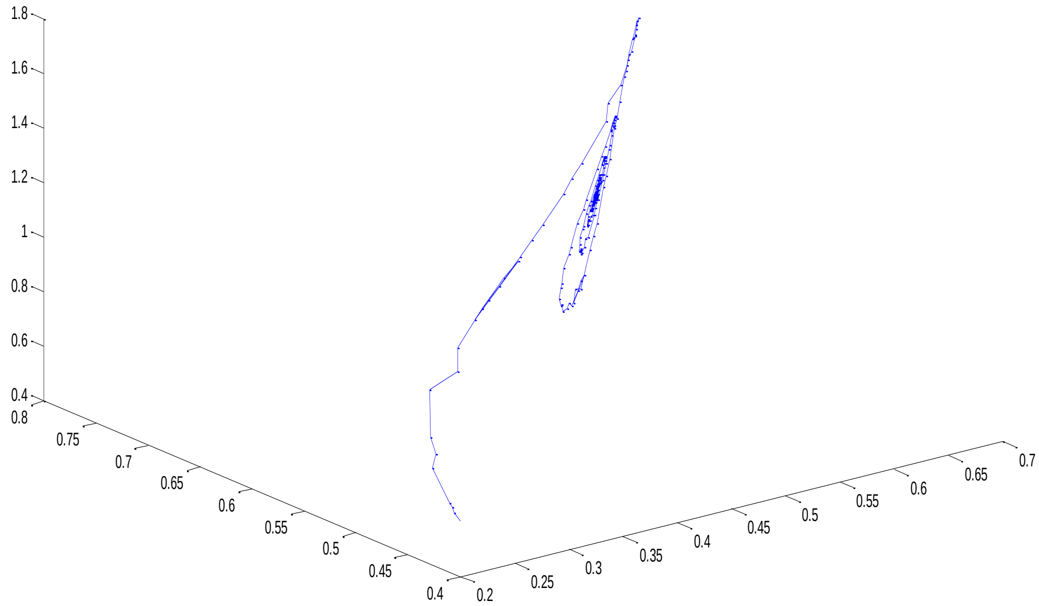

Figure 5.1 Phase plane graph for asymptotically stable when $\tau < \tau_0 = 1.7387$

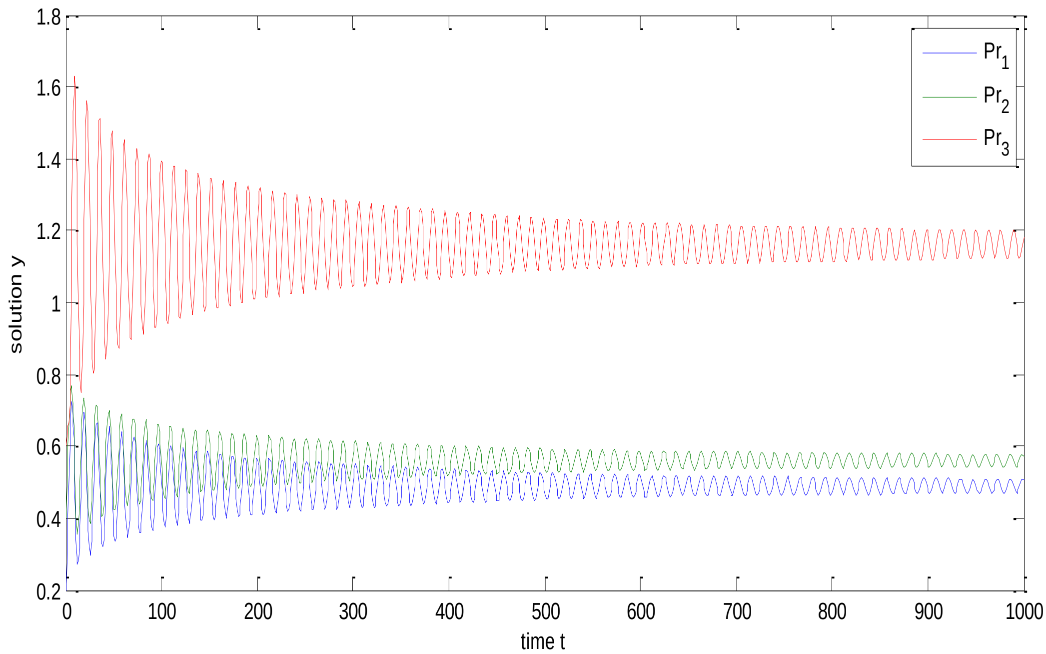

Figure 6. Hopf Bifurcation when $\tau = 2.5 > \tau_0 = 1.7387$

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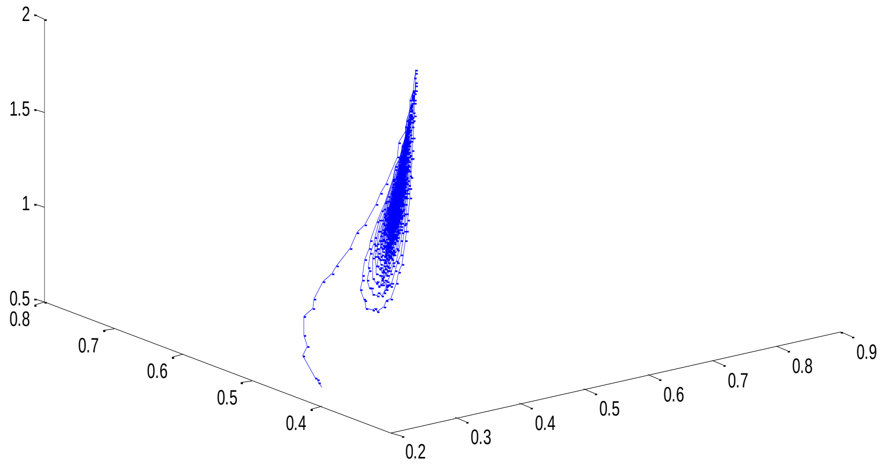

Figure 6.1 Phase plane graph for Hopf bifurcation when $\tau = 2.5 > \tau_0 = 1.7387$

5. Conclusion

Certain species, such as zebras and gazelles, form teams because it reduces the predation risk. Time delay caused by the age structure, maturation period, and feeding time is a major factor in real-time prey–predator dynamic that results in periodic solutions and the bifurcation phenomenon. This study investigated the impact of lag time on a multiteam prey–predator dynamic by examining two prey and one predator and considering that the two prey populations support each other when they are susceptible to predation. The insertion of a time delay destabilizes the system's stable equilibrium point. For set 1, the system is absolutely stable in the absence of delay (i.e, $\tau = 0$; Figure 1). The same finding is analytically supported by Ruth–Hurwitz's criteria. The system is asymptotically stable when the value of delay is less than the critical value (i.e., $\tau < 1.7387$; Figures 2 and 2.1). The Hopf bifurcation is observed when the delay parameter passes a critical value (i.e., $\tau \geq 1.7387$; Figures 3 and 3.1). For set 2, the system is absolutely stable in the absence of delay (i.e., $\tau = 0$; Figures 4, 4.1, and 4.2). The same finding is analytically supported by Ruth–Hurwitz's criteria. The system is asymptotically stable when the value of delay is less than the critical value (i.e., $\tau < 1.7387$; Figures 5 and 5.1). The Hopf bifurcation is observed when the delay parameter passes a critical value (i.e., $\tau \geq 1.7387$; Figures 6 and 6.1). These graphs have their basics covered in lemmas 1 and 2. Furthermore, the technique used to determine the direction and stability of a Hopf bifurcation solution is constructed using normal form theory and the centre manifold reduction hypothesis. Numerical results are substantiated using the dde23 code of MATLAB

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