A Two Parameter Odd Exponentiated Skew-T Distribution With J-Shaped Hazard Rate Function

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Abstract

A new generalization of the skew-t distribution was proposed. The two-parameter lifetime model called the odd exponentiated skew-t distribution has the ability of fitting skewed, long and heavy tailed datasets. It is considered to be more flexible than the skew-t distribution as it contains it as a special case. Some basic properties of the distribution such as the order statistics, entropy, asymptotic behaviour, moment, incomplete moment, characteristic function and quantile function were derived. The odd exponentiated skew-t distribution parameter estimates were derived using the maximum likelihood estimation method and simulation studies performed to evaluate the finite sample performance of these parameter estimates showed that the parameter estimates were consistent and approached the arbitrary selected parameter values as the sample size is increased. The application using a real-life dataset indicated that the new distribution outperformed the other competing distributions. The hazard rate shape of the odd exponentiated skew-t distribution was found to be increasing and J-shaped which was also reflected in the application result.

Keywords: Maximum likelihood estimation, Moments, Order statistics, Skew-t distribution, Odd exponentiated generator.

1. INTRODUCTION

The methods of extending the flexibility of various continuous probability distributions are well-known in the literature. Hence, significant efforts in developing new families of flexible continuous probability distributions and extending the efficacy of the existing distributions have been made by several authors over the years due to inability of the classical distributions to fit various real-life datasets. The skew-t distribution introduced as an extension of the symmetric t-distribution has been used extensively especially in the field of econometric, time series and financial analysis. Numerous authors have introduced various complex forms of the skew-t which lacked a defined expression for the density function, example of the forms can be found in Johnson *et al.* (1995), Azzalini and Capitanio (2003), Sahu *et al.* (2003) and Gupta (2002). Several authors have studied possible extensions and generalizations of the skew-t distribution: Shafiei and Doostparast (2014) proposed a new generalization of the skew-t distribution of Azzalini and Capitanio (2003) called the Balakrishnan skew-t (BST) distribution, as a scale mixture of the Balakrishnan normal distribution. The density function shape of the BST is right-skewed at different degree of freedom which gives it more flexibility in fitting skewed datasets. Aas and Haff (2006) proposed the generalized hyperbolic skew-t (GHST) distribution which is considered as a limiting case of the generalized hyperbolic (GH) distribution. They stated that the generalized hyperbolic skew-t (GHST) distribution can be represented as a mixing distribution comprising normal variance-mean mixture with the generalized inverse gaussian distribution. Khamis *et al.* (2017) proposed the Kumaraswamy skew-t (KwST) distribution which has the ability of fitting heavy-tailed and skewed datasets than the skew-t distribution of Azzalini and Capitanio (2014). Basalamah *et al.* (2018) introduced a new generalization of the skew-t distribution of Azzalini and Capitanio (2014) called the Beta skew-t (BST) distribution. The maximum likelihood and L-moments methods were used in demonstrating the flexibility of the BST distribution in fitting real datasets and the results were in favour of the BST distribution. These presented extensions with a lot of parameters were based on the complex skew-t distribution.

The noncomplex one-parameter tractable skew-t distribution introduced by Jones and Faddy (2003) with defined density and distribution functions was established by introducing a scaling factor into the two degrees of freedom of the simplest student-t form derived by Jones (2002). The main aim of this article is to introduce a new hybridized distribution with fewer parameters, with the expectation it produces a better fit in certain real-world situations and in a wider range of real-life datasets in engineering, biology, medicine and finance. Additionally, a complete derivation of the statistical properties of the proposed distribution are provided. The purpose for developing the two-parameter hybridized distribution is to furnish a more flexible distribution with skewed and unimodal features that can handle properly skewed and leptokurtic real datasets often found in various fields better than existing two-parameter distributions. This new distribution in this article can find its potentiality as an alternative conditional error distribution in GARCH framework when used in volatility modeling. The rest of the paper is organized as follows. In Section 2, introduce the new distribution called the odd exponentiated skew-t (OE_{ST}) distribution. In Section 3, statistical properties of the proposed distribution are derived. In section 4, the maximum likelihood estimation method is applied to derive the estimates of the model parameters and simulation study performed to assess the performance of the OE_{ST} parameter estimates. In section 5, a dataset application is illustrated to demonstrate the superiority of the new distribution while section 6 concludes the study.

2. ODD EXPONENTIATED SKEW-T DISTRIBUTION

Jones (2001) and Jones and Faddy (2003) established a tractable skewed extension of the symmetric student-t distribution known as the skew student-t (skew-t) distribution. The cumulative distribution function (CDF) is given as:

$$G_{ST}(y) = \frac{1}{2}\left(1 + \frac{y}{\sqrt{\lambda + y^2}}\right), \quad y \in (-\infty, \infty) \tag{1}$$

The probability distribution function (PDF) obtained by differentiating (1) is given as

$$g_{ST}(y) = \frac{\lambda}{2\left(\lambda + y^2\right)^{3/2}} \tag{2}$$

where λ is the skew parameter.

The odd exponentiated family of distributions is a special case established by setting $\beta = 1$in the density and distribution functions of the Weibull-G family (Bourguignon *et al.*, 2014). The CDF is given by

$$F(y) = \left\{1 - e^{-\alpha \frac{G(y)}{1-G(y)}}\right\}, \tag{3}$$

The PDF by differentiating (3) is given as:

$$f(y) = \frac{\alpha g(y)}{\left[1 - G(y)\right]^2} e^{-\alpha \frac{G(y)}{1-G(y)}}, \tag{4}$$

where $\alpha > 0$ is the shape parameter, $G(y)$ and $g(y)$ are the baseline distribution CDF and PDF.

A two-parameter model called the odd exponentiated skew-t (OE_{ST}) distribution is introduced. The PDF is obtained by inserting Equations (1) and (2) into Equation (4) expressed as:

$$f(y;\upsilon)=\frac{\alpha\lambda}{2\left(\lambda+y^2\right)^{3/2}\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)\right)\right]^2}e^{-\alpha\frac{\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)\right)}{1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)\right)}},\quad y\in(-\infty,\infty);\alpha,\lambda>0, \tag{5}$$

The corresponding CDF by inserting (1) in (3) is given as:

$$F(y;\upsilon)=1-e^{-\alpha\frac{\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)\right)}{1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)\right)}} \tag{6}$$

From now onward, a random variable *Y* having PDF (5) is denoted by $OE_{ST}(\upsilon)$, where $\upsilon=(\alpha,\lambda)$ are set of parameters.

The survival function is defined as $s(y)=1-F(y)$, given a random variable *Y*. Hence, the survival function $s(y)$ of OE_{ST} distribution is given as:

$$S(y)=e^{-\alpha\frac{\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)\right)}{1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)\right)}} \tag{7}$$

The hazard rate function $h(y)$ is given as:

$$h(y)=\frac{\alpha\lambda}{2\left(\lambda+y^2\right)^{3/2}\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)\right)\right]^2} \tag{8}$$

To show the efficacy of the OE_{ST} distribution, the graphical structures of the OE_{ST} density function and distribution function are depicted in Figures 1 and 2 with the skew parameter λ kept constant and the shape parameter α varied. Figure 1, indicates that the right tail of the density function gets lighter and tend to zero as α approach infinity. More so, Figure 2 indicates that the shape of the CDF is within the limits of zero and one, which justifies that the proposed OE_{ST} distribution is a valid distribution.

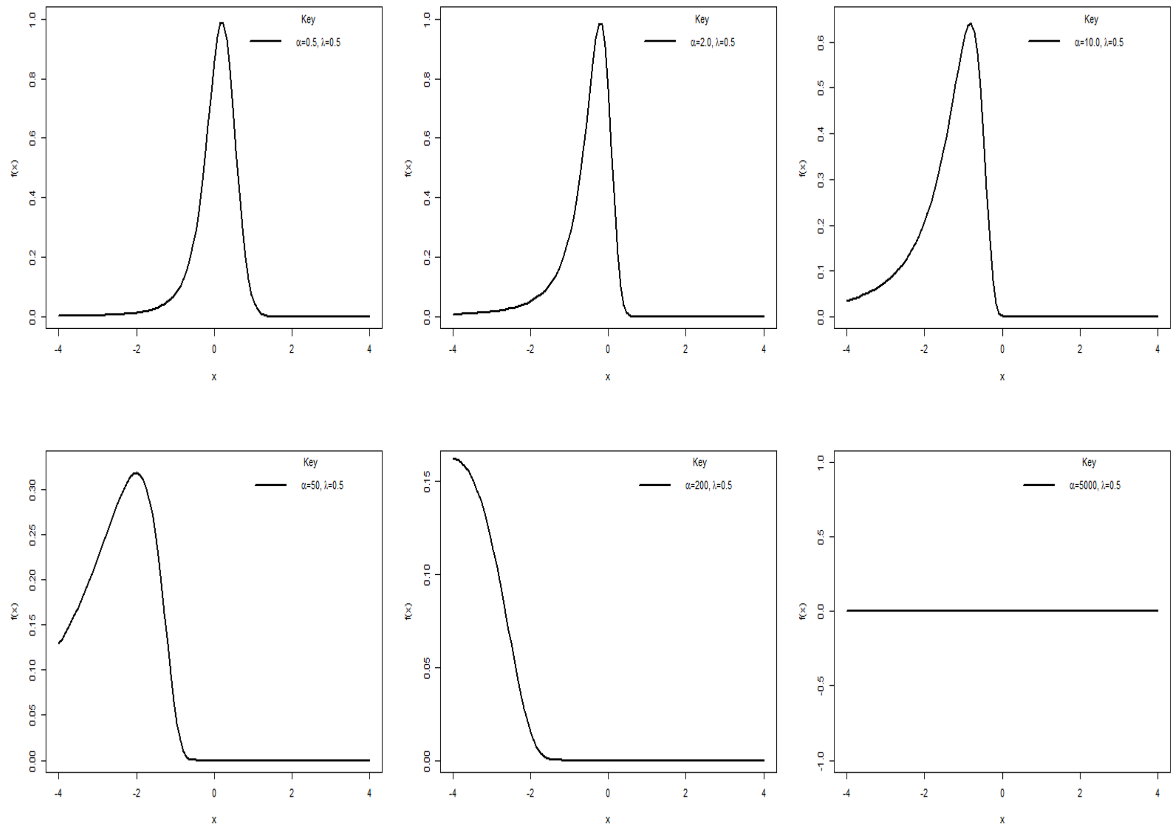

Figure 1: The OE_{ST} density function plots for some selected $(\alpha = \text{varied}, \lambda = 0.5)$ parameter values.

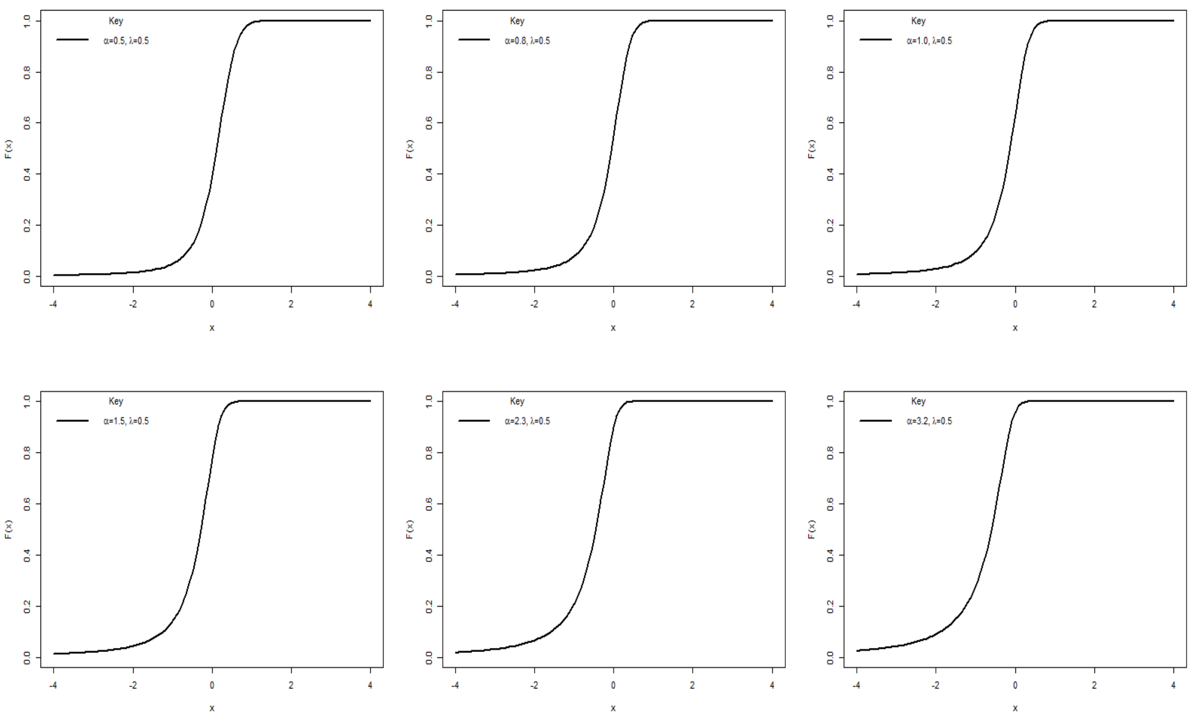

Figure 2: The OE_{ST} distribution function plots for some selected $(\alpha = \text{varied}, \lambda = 0.5)$ parameter values.

Likewise, the shapes of the hazard rate function depicted in Figure 3, reveal that it can be increasing and J-shaped.

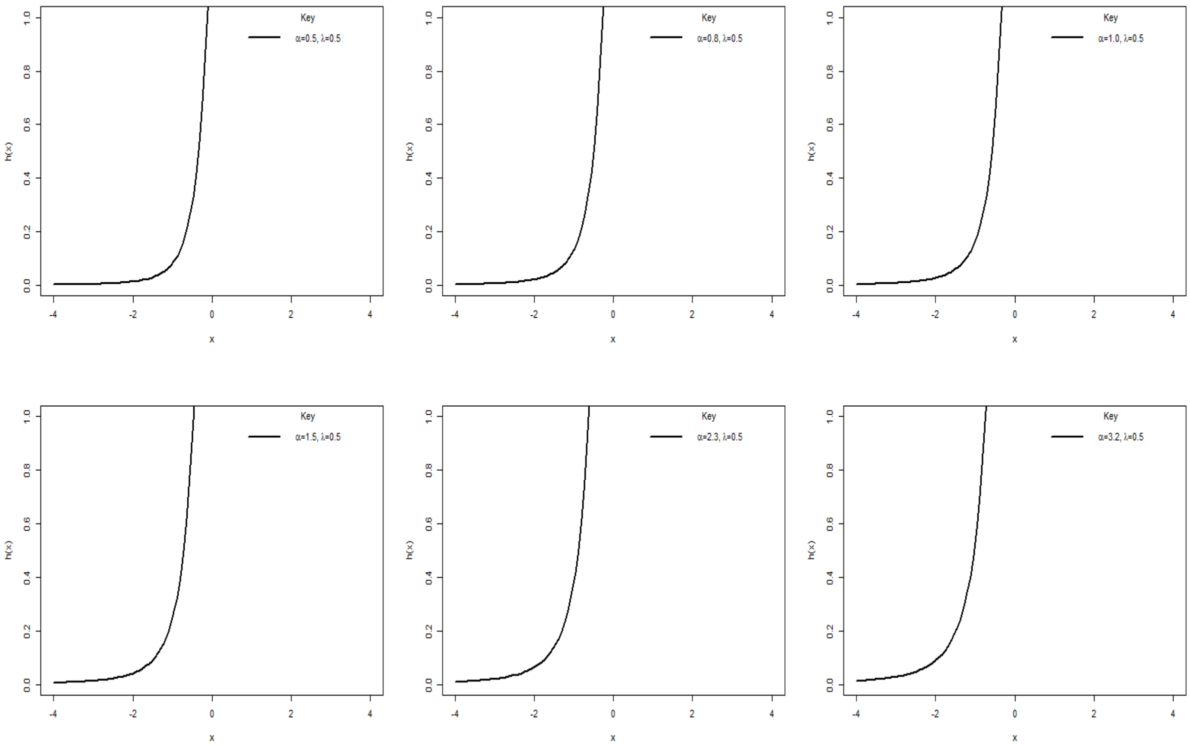

Figure 3: The OE_{ST} hazard rate function plots for some selected $(\alpha = \text{varied}, \lambda = 0.5)$ parameter values.

3. STATISTICAL PROPERTIES

In this section, some basic statistical properties of the OE_{ST} distribution are derived.

3.1 Quantile Function

The quantile function $Q(u) = F(y)^{-1}$ for $u \in (0,1)$ of the OE_{ST} distribution is given by:

$$Q(u) = \frac{\lambda^{1/2}\left(2\left\langle \dfrac{-\log(1-u)}{(\alpha - \log(1-u))} \right\rangle - 1\right)}{\left(1 - \left(2\left\langle \dfrac{-\log(1-u)}{(\alpha - \log(1-u))} \right\rangle - 1\right)^2\right)^{1/2}}, \qquad u \in (0,1). \tag{9}$$

The median $Q(0.5)$ is derived by setting $u = 0.5$ in (9). The other quantiles can be derived similarly by setting $u = 0.25$ and $u = 0.75$.

$$Q(0.5)=\frac{\lambda^{1/2}\left(2\left\langle\frac{-\log(0.5)}{(\alpha-\log(0.5))}\right\rangle-1\right)}{\left(1-\left(2\left\langle\frac{-\log(0.5)}{(\alpha-\log(0.5))}\right\rangle-1\right)^{2}\right)^{1/2}}, \qquad u\in(0,1). \tag{10}$$

The OE_{ST} quantile function (9) can be used in generating random values from the OE_{ST} distribution. The Bowley skewness (Kenney & Keeping,1962) and Moors kurtosis (Moors, 1988) are as follows:

$$S_k=\frac{Q\left(\frac{3}{4};\varphi,\lambda\right)-2Q\left(\frac{1}{2};\varphi,\lambda\right)+Q\left(\frac{1}{4};\varphi,\lambda\right)}{Q\left(\frac{3}{4};\varphi,\lambda\right)-Q\left(\frac{1}{4};\varphi,\lambda\right)} \tag{11}$$

$$K=\frac{Q\left(\frac{7}{8};\varphi,\lambda\right)-Q\left(\frac{5}{8};\varphi,\lambda\right)-Q\left(\frac{3}{8};\varphi,\lambda\right)+Q\left(\frac{1}{8};\varphi,\lambda\right)}{Q\left(\frac{6}{8};\varphi,\lambda\right)-Q\left(\frac{2}{8};\varphi,\lambda\right)} \tag{12}$$

where $Q(.)$ represent the quantile function. Using the OE_{ST} quantile function (9), the numeric values of the median (M), 25th and 75th percentiles, interquartile range (IQR), kurtosis (Ks), and skewness (Sk) for some chosen parameter values are provided in Table 1. It is clear from Table 1, as the values of λ increases at specific values of α, the skewness and kurtosis remain constant. More so, across different values of α, the skewness and kurtosis decreases indicating negative properties, respectively.

Table 1: Descriptive statistics of the OE_{ST} distribution.

α	λ	**Ks**	**Sk**	**M**	**25th**	**75th**	**IQR**
0.2	0.3	-0.0511	-0.0161	0.3627	0.1001	0.6170	0.5169
	0.5	-0.0511	-0.0161	0.4683	0.1292	0.7965	0.6673
	0.9	-0.0511	-0.0161	0.6283	0.1734	1.0687	0.8953
	1.2	-0.0511	-0.0161	0.7254	0.2002	1.2340	1.0338
	1.5	-0.0511	-0.0161	0.8111	0.2238	1.3796	1.1558
0.4	0.3	-0.1616	-0.0725	0.1525	-0.0907	0.3627	0.4534
	0.5	-0.1616	-0.0725	0.1968	-0.1171	0.4683	0.5853
	0.9	-0.1616	-0.0725	0.2641	-0.1571	0.6283	0.7853
	1.2	-0.1616	-0.0725	0.3049	-0.1813	0.7254	0.9068
	1.5	-0.1616	-0.0725	0.3409	-0.2028	0.8111	1.0138
0.6	0.3	-0.2291	-0.1106	0.0396	-0.2059	0.2361	0.4420

	0.5	-0.2291	-0.1106	0.0511	-0.2658	0.3048	0.5706
	0.9	-0.2291	-0.1106	0.0685	-0.3658	0.4089	0.7655
	1.2	-0.2291	-0.1106	0.0791	-0.4117	0.4722	0.8840
	1.5	-0.2291	-0.1106	0.0884	-0.4603	0.5280	0.9883
0.7	0.3	-0.2539	-0.1253	-0.0027	-0.2516	0.1908	0.4424
	0.5	-0.2539	-0.1253	-0.0035	-0.3248	0.2463	0.5712
	0.9	-0.2539	-0.1253	-0.0047	-0.4358	0.3305	0.7663
	1.2	-0.2539	-0.1253	-0.0054	-0.5032	0.3816	0.8848
	1.5	-0.2539	-0.1253	-0.0060	-0.5626	0.4266	0.9892
1.5	0.3	-0.3599	-0.1935	-0.2167	-0.5054	-0.0216	0.4838
	0.5	-0.3599	-0.1935	-0.2797	-0.6525	-0.0279	0.6246
	0.9	-0.3599	-0.1935	-0.3753	-0.8754	-0.0374	0.8380
	1.2	-0.3599	-0.1935	-0.4334	-1.0108	-0.0432	0.9676
	1.5	-0.3599	-0.1935	-0.4846	-1.1301	-0.0483	1.0818

3.2 Asymptotic Behaviour

The limits of the OE_{ST} density function (PDF) are given by:

$$\lim_{y\to-\infty} f(y) = \lim_{y\to+\infty} f(y) = 0$$

Proof:

For $y \to -\infty$, we have

$$\lim_{y\to-\infty} f(x) = \lim_{y\to-\infty}\left(\frac{\alpha\lambda}{2\left(\lambda+y^2\right)^{3/2}\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)\right)\right]^2} e^{-\alpha\frac{\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)\right)}{1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)\right)}}\right) \tag{13}$$

It is obvious that $\lim_{x\to-\infty}\left(\frac{\lambda}{2\left(\lambda+x^2\right)^{3/2}}\right) = 0$.

Therefore, $\lim_{y\to-\infty} f(x) = 0$ (14)

Similarly, for $y \to +\infty$, we have

$$\lim_{y\to+\infty} f\left(x\right) = \lim_{y\to+\infty}\left(\frac{\alpha\lambda}{2\left(\lambda+y^2\right)^{3/2}\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)\right)\right]^2} e^{-\alpha\frac{\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)\right)}{1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)\right)}}\right) \quad (15)$$

It is obvious that $\lim_{x\to+\infty}\left(\frac{\lambda}{2\left(\lambda+x^2\right)^{3/2}}\right)=0$.

Therefore, $\lim_{y\to+\infty} f\left(x\right)=0$ (16)

The results of the asymptotic behaviour infer that the OE_{ST} mode is unique.

3.3 Mixture Representations

The series expansion of the OE_{ST} distribution is derived for the density and cumulative functions. This mixture representation is important to derive several statistical properties of this distribution in full generality. If $|s|<1$ and k a positive real non-integer, the generalized binomial theorem representation is given by:

$$\left(1-s\right)^{k-1} = \sum_{j=0}^{\infty}(-1)^j\binom{k-1}{j}s^j \quad (17)$$

The expanded form of the PDF, applying the series expansion in Equation (17) in (5) leads to:

$$f(y) = \alpha\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}\,\alpha^i}{i!}\binom{-(i+2)}{j}\left(\frac{\lambda}{2\left(\lambda+y^2\right)^{3/2}}\right)\left[\frac{1}{2}\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)\right]^{(i+j)} \quad (18)$$

The preceding equation reveals that the PDF expression is likely an infinite linear combination of the skew-t density functions. Thus, we can obtain the statistical properties of the OE_{ST} distribution from the properties of the skew-t distribution. Also, another expanded form of the PDF is given by

$$f(y) = w_{i,j,k,l,m}\,y^m\left(\lambda+y^2\right)^{-\left(\frac{m+3}{2}\right)} \quad (19)$$

where $w_{i,j,k,l,m} = \alpha\lambda\frac{1}{2^{l+1}}\sum_{i,j,k,l=0}^{\infty}\sum_{m=0}^{l}\frac{(-1)^{i+j+k+l}\,\alpha^i}{i!}\binom{-(i+2)}{j}\binom{i+j}{l}\binom{k}{l}\binom{l}{m}$.

The expanded form of the CDF of the OE_{ST} distribution by applying series expansion to Equation (6), is given by

$$F\left(y\right) = \psi_{i,j,k,l,m,q}\,x^q\left(\lambda+y^2\right)^{-\frac{q}{2}} \quad (20)$$

where $\psi_{i,j,k,l,m,q} = \frac{1}{2^m} \sum_{i,j,k,l,m=0}^{\infty} \sum_{q=0}^{m} \frac{(-1)^{i+j+k+l+m} (i\alpha)^j}{j!} \binom{j}{k} \binom{j+k}{l} \binom{l}{m} \binom{m}{q}$

3.4 Moments

Let $Y \sim OE_{ST}(\alpha, \lambda)$ be a random variable, then the g[th] moment of Y is given by

$$\mu_g' = \int_{-\infty}^{+\infty} y^g w_{i,j,k,l,m} y^m \left(\lambda + y^2\right)^{-\left(\frac{m+3}{2}\right)} dy \tag{21}$$

Taboga (2017, p.413, s.50.1.5) showed that (21) can be rewritten as:

$$\mu_g' = \left(1 + (-1)^g\right) w_{i,j,k,l,m} \int_0^{+\infty} y^{g+m} \left(\lambda + y^2\right)^{-\left(\frac{m+3}{2}\right)} dy \tag{22}$$

After some algebra, using the Beta function expression $B(\theta,\gamma) = \int_0^{+\infty} y^{\theta-1} (1+y)^{-\theta-\gamma} dy$. The r[th] moment is given by

$$\mu_g' = \begin{cases} w_{i,j,k,l,m} \lambda^{\frac{g-2}{2}} \mathrm{B}\left(\frac{g+m+1}{2}, \frac{2-g}{2}\right) & g = even \\ 0 & g = odd \end{cases} \tag{23}$$

where $w_{i,j,k,l,m} = \alpha\lambda \frac{1}{2^{l+1}} \sum_{i,j,k,l=0}^{\infty} \sum_{m=0}^{l} \frac{(-1)^{i+j+k+l} \alpha^i}{i!} \binom{-(i+2)}{j} \binom{i+j}{l} \binom{k}{l} \binom{l}{m}$

The incomplete moment of the OE_{ST} distribution is derived. Let $Y \sim OE_{ST}(\alpha, \lambda)$ be a random variable, the g[th] incomplete moment for any $t > 0$ is given by

$$\varphi_g'(t) = \int_0^t y^g w_{i,j,k,l,m} y^m \left(\lambda + y^2\right)^{-\left(\frac{m+3}{2}\right)} dy \tag{24}$$

After some algebra, using the Beta function expression $B(z,\theta,\gamma) = \int_0^z y^{\theta-1} (1-y)^{\gamma-1} dy$. The r[th] incomplete moment is given by

$$\varphi_g'(t) = w_{i,j,k,l,m} \lambda^{\frac{g-2}{2}} \mathrm{B}\left(t, \frac{g+m+1}{2}, \frac{2-g}{2}\right) \tag{25}$$

where $w_{i,j,k,l,m} = \alpha\lambda \frac{1}{2^{l+1}} \sum_{i,j,k,l=0}^{\infty} \sum_{m=0}^{l} \frac{(-1)^{i+j+k+l} \alpha^i}{i!} \binom{-(i+2)}{j} \binom{i+j}{l} \binom{k}{l} \binom{l}{m}$

Remark: The first incomplete moment $\varphi_1'(t) = \int_0^t y f(y) dy$ of OE_{ST} distribution can be obtained by inserting $g = 1$ in (25).

3.5 Characteristics Function

The characteristics function of a random variable Y is a function $\varphi_X(t)$ defined as:

$$\varphi_X(t) = \mathrm{E}\left(e^{itX}\right) = \sum_{g=0}^{\infty} \frac{(it)^g}{g!}\mu_g' \tag{26}$$

Inserting Equation (23) into Equation (26), the characteristics function of the odd exponentiated skew-t (OE_{ST}) distribution is given as:

$$\varphi_X(t) = \sum_{g=0}^{\infty} \frac{(it)^g}{g!} \begin{cases} w_{i,j,k,l,m}\lambda^{\frac{g-2}{2}} \mathrm{B}\left(\frac{g+m+1}{2}, \frac{2-g}{2}\right) & g = even \\ 0 & g = odd \end{cases} \tag{27}$$

where $w_{i,j,k,l,m} = \alpha\lambda \frac{1}{2^{l+1}} \sum_{i,j,k,l=0}^{\infty} \sum_{m=0}^{l} \frac{(-1)^{i+j+k+l}\alpha^i}{i!} \binom{-(i+2)}{j} \binom{i+j}{l} \binom{k}{l} \binom{l}{m}$

3.6 Order Statistics

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a continuous distribution and $Y_{1:n} < Y_{2:n} < \ldots < Y_{n:n}$ are the order statistics obtained from the sample. The r[th] order statistic $Y_{r:n}$ is defined as

$$f_{r:n}(y) = \frac{1}{\mathrm{B}(r, n-r+1)} \sum_{l=0}^{n-r} (-1)^l \binom{n-r}{l} \left[G(y)\right]^{r+l-1} g(y) \tag{28}$$

Inserting Equations (5) and (6) in Equation (28), applying series expansion. The r[th] order statistics for OE_{ST} distribution is given as

$$f_{r:n}(y) = \frac{1}{\mathrm{B}(p, n-p+1)} \vartheta_{l,i,j,k,g,h,m} y^m \left(\lambda + y^2\right)^{-\left(\frac{m+3}{2}\right)} \tag{29}$$

where $\vartheta_{l,i,j,k,g,h,m} = \frac{1}{2^{h+1}} \alpha\lambda \sum_{l=0}^{n-p} \sum_{i,j,k,g,h=0}^{\infty} \sum_{m=0}^{h} \frac{(-1)^{i+j+k+l+g+h} \left(\alpha(i+1)\right)^j}{j!} \binom{p+l-1}{i} \binom{-(j+2)}{k} \binom{n-p}{l} \binom{j+k}{g} \binom{g}{h} \binom{h}{m}$

Remark: The minimum and maximum order statistics is derived by setting $r = 1$ and $r = n$ in Equation (29).

3.7 Entropies

The variation of uncertainty in a random variable is normally measured by the entropy (Rényi, 1961). The Rényi entropy $\mathrm{I}_{R(\delta)}$ is expressed as:

$$\mathrm{I}_{R(\delta)} = \frac{1}{1-\delta} \log \int_{-\infty}^{+\infty} f(y)^{\delta} dy, \qquad \delta > 0 \text{ and } \delta \neq 1 \tag{30}$$

Using the PDF mixture representation of OE_{ST} distribution in (19), $f(y)^{\delta}$ is given as:

$$f(y)^{\delta} = w_{i,j,k,l,m} y^{m} \left(\lambda + y^{2}\right)^{-\left(m+3\delta/2\right)} \tag{31}$$

where $w_{i,j,k,l,m} = \frac{(\alpha\lambda)^{\delta}}{2^{l+\delta}} \sum_{i,j,k,l=0}^{\infty} \sum_{m=0}^{l} \frac{(-1)^{i+j+k+l} (\alpha\delta)^{i}}{i!} \binom{-(i+2\delta)}{j} \binom{i+j}{k} \binom{k}{l} \binom{l}{m}$

Hence, the Rényi entropy of the OE_{ST} distribution using the expression by Taboga (2017, p.413, s.50.1.5), is expressed as:

$$\mathrm{I}_{R(\delta)} = \frac{1}{1-\delta} \log\left(\left(1 + (-1)^{g}\right) w_{i,j,k,l,m} \int_{0}^{+\infty} y^{m} \left(\lambda + y^{2}\right)^{-\left(m+3\delta/2\right)} dy \right) \tag{32}$$

Using the expression of the Beta function $B(\theta,\gamma) = \int_{0}^{+\infty} y^{\theta-1} (1+y)^{-\theta-\gamma} dy$. The Rényi entropy of the OE_{ST} distribution is given as:

$$\mathrm{I}_{R(\delta)} = \frac{1}{1-\delta} \log \begin{cases} w_{i,j,k,l,m} \lambda^{\frac{1-3\delta}{2}} \mathrm{B}\left(\frac{m+1}{2}, \frac{3\delta-1}{2}\right) & g = even \\ 0 & g = odd \end{cases} \tag{33}$$

Likewise, the q-entropy (Tsallis, 1988) is defined as:

$$\mathrm{H}_{q} = \frac{1}{q-1} \log\left(1 - \int_{-\infty}^{+\infty} f(y)^{q} dy\right), \; q > 0 \text{ and } q \neq 0 \tag{34}$$

Using the PDF mixture representation of OE_{ST} distribution in (19), $f(y)^{q}$ is given as:

$$f(y)^{q} = w_{i,j,k,l,m} y^{m} \left(\lambda + y^{2}\right)^{-\left(m+3q/2\right)} \tag{35}$$

where $w_{i,j,k,l,m} = \frac{(\alpha\lambda)^{q}}{2^{l+q}} \sum_{i,j,k,l=0}^{\infty} \sum_{m=0}^{l} \frac{(-1)^{i+j+k+l} (\alpha q)^{i}}{i!} \binom{-(i+2q)}{j} \binom{i+j}{k} \binom{k}{l} \binom{l}{m}$

Hence, the q-entropy of the OE_{ST} distribution is given as:

$$\mathrm{H}_{q} = \frac{1}{q-1} \log\left\{ 1 - \left(\begin{cases} w_{i,j,k,l,m} \lambda^{\frac{1-3q}{2}} \mathrm{B}\left(\frac{m+1}{2}, \frac{3q-1}{2}\right) & g = even \\ 0 & g = odd \end{cases} \right) \right\} \tag{36}$$

4 MODEL ESTIMATION

4.1 Parameters Estimation

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from the OE_{ST} distribution with unknown parameter vector $\upsilon = (\alpha, \lambda)^{T}$. The likelihood (L) of the OE_{ST} distribution is expressed as:

$$L(\upsilon)=\prod_{i=1}^{n}\left(\frac{\alpha\lambda}{2\left(\lambda+y^2\right)^{3/2}\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)\right)\right]^2}e^{-\alpha\frac{\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)\right)}{1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)\right)}}\right) \tag{37}$$

The log-likelihood function (LL) is given as:

$$LL=\log L(\upsilon)=n\log\alpha+n\log\lambda-n\log 2-3/2\sum_{i=1}^{n}\log\left(\lambda+y^2\right)-\alpha\sum_{i=1}^{n}\frac{\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)\right)}{1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+y^2}}\right)\right)} \tag{38}$$
$$-2\sum_{i=1}^{n}\log\left(1-\frac{1}{2}\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)\right)$$

Taking the partial derivative of the log-likelihood l, with respect to α and λ equating to zero, the normal equations are obtained as follows:

$$\frac{\partial LL}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\frac{\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+y^2}}\right)\right)}{1-\left(\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+y^2}}\right)\right)}=0 \tag{39}$$

$$\frac{\partial LL}{\partial\lambda}=\frac{n}{\lambda}-\frac{3}{2}\sum_{i=1}^{n}\frac{1}{\left(\lambda+y^2\right)}+\alpha\sum_{i=1}^{n}\frac{y}{\sqrt{\lambda+y^2}\left(\sqrt{\lambda+y^2}-y\right)^2}+\sum_{i=1}^{n}\frac{y}{\left(\lambda+y^2\right)\left(-\sqrt{\lambda+y^2}+y\right)}=0 \tag{40}$$

The non-linear equations (38) and (39) are solved numerically via iterative methods using statistical software such as R, MATLAB and Maple. The maximum likelihood estimates (MLEs) are asymptotic normally distributed i.e., $\sqrt{n}\left(\hat{\alpha}-\alpha,\hat{\lambda}-\lambda\right)$ follow $\mathrm{N}_2(0,\Sigma)$, where Σ is the variance-covariance matrix obtained by inverting the observed Fisher information (F) given as follows:

$$F=\begin{bmatrix}\frac{\partial^2 l}{\partial\alpha^2} & \frac{\partial^2 l}{\partial\alpha\partial\lambda}\\ \frac{\partial^2 l}{\partial\alpha\partial\lambda} & \frac{\partial^2 l}{\partial\lambda^2}\end{bmatrix}$$

For each parameter of OE_{ST} distribution, the asymptotic $(1-\tau)100\%$ confidence intervals are estimated with

$$\hat{\alpha} \pm Z_{\tau/2}\sqrt{\Sigma_{11}}$$

$$\hat{\lambda} \pm Z_{\tau/2}\sqrt{\Sigma_{22}}$$

where, upper τ^{th} percentile of the standard normal distribution is Z_{τ}.

4.2 Simulations Study

The simulation study for the average MLEs, absolute bias, variance, Mean Square Errors (MSE), and Root Mean Square Errors (RMSE) are performed for the OE_{ST} distribution. For $N = 10000$, random numbers of sample size $n = 30, 50, 150, 300, 1000$ are generated using the OE_{ST} quantile function as in equation (9). The absolute bias, MSE and RMSE are computed for $\hat{S} = (\hat{\alpha}, \hat{\lambda})$ using

$$AbsBias_s = \left| \frac{1}{N}\sum_{i=1}^{N}\left(\hat{S}_i - S\right) \right|, \qquad MSE_s = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{S}_i - S\right)^2, \qquad RMSE_s = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\hat{S}_i - S\right)^2}$$

The simulation results for the average MLEs, absolute bias, variance, MSEs, and RMSEs for different combinations of the parameters α and λ are given in Table 2. These estimates are sensibly consistent and approach the parameter values as the sample size increases. The absolute bias, variance, RMSEs, and MSEs decrease for all parameter mixtures as the sample size increases which implies that the OE_{ST} parameter estimates are very much consistent, better estimated and approaches the arbitrary selected parameter values as the sample size increases.

Table 2: Simulation results

$(\alpha = 1.2, \lambda = 0.7)$						
n	**Par**	**AE**	**ABS**	**Var**	**MSE**	**RMSE**
30	α	1.241	0.041	0.064	0.066	0.256
	λ	0.687	0.013	0.061	0.061	0.248
50	α	1.222	0.022	0.035	0.035	0.187
	λ	0.691	0.009	0.037	0.037	0.193
150	α	1.206	0.006	0.101	0.011	0.103
	λ	0.696	0.004	0.012	0.012	0.110
300	α	1.203	0.003	0.005	0.005	0.072
	λ	0.698	0.002	0.006	0.006	0.077
1000	α	1.201	0.000	0.001	0.001	0.039
	λ	0.700	0.000	0.002	0.002	0.042

$(\alpha = 1.5, \lambda = 1.0)$						
n	**Par**	**AE**	**ABS**	**Var**	**MSE**	**RMSE**
30	α	1.570	0.070	0.105	0.110	0.332
	λ	0.978	0.022	0.130	0.131	0.362
50	α	1.539	0.039	0.055	0.056	0.237
	λ	0.985	0.015	0.079	0.079	0.281
150	α	1.510	0.101	0.016	0.016	0.127
	λ	0.994	0.006	0.025	0.025	0.160
300	α	1.506	0.006	0.008	0.008	0.088
	λ	0.997	0.003	0.012	0.012	0.112
1000	α	1.501	0.001	0.002	0.002	0.048
	λ	0.999	0.001	0.004	0.004	0.061

$(\alpha = 1.7, \lambda = 1.2)$						
n	**Par**	**AE**	**ABS**	**Var**	**MSE**	**RMSE**
30	α	1.795	0.095	0.147	0.156	0.395
	λ	1.170	0.030	0.195	0.195	0.442
50	α	1.752	0.052	0.074	0.077	0.277
	λ	1.180	0.020	0.118	0.118	0.344
150	α	1.714	0.014	0.021	0.021	0.146
	λ	1.192	0.007	0.038	0.038	0.195
300	α	1.708	0.008	0.010	0.010	0.100
	λ	1.196	0.004	0.019	0.019	0.136
1000	α	1.702	0.002	0.003	0.003	0.055
	λ	1.199	0.001	0.005	0.005	0.074

$(\alpha = 2.0, \lambda = 1.5)$						
n	**Par**	**AE**	**ABS**	**Var**	**MSE**	**RMSE**
30	α	2.141	0.141	0.244	0.264	0.514
	λ	1.457	0.043	0.324	0.326	0.571
50	α	2.078	0.078	0.115	0.121	0.348
	λ	1.471	0.029	0.195	0.196	0.443
150	α	2.022	0.022	0.031	0.031	0.177
	λ	1.489	0.010	0.063	0.063	0.251
300	α	2.011	0.011	0.014	0.015	0.121
	λ	1.495	0.005	0.031	0.031	0.176
1000	α	2.003	0.003	0.004	0.004	0.066
	λ	1.499	0.001	0.009	0.009	0.095

5 APPLICATION

In this section, the flexibility and superiority of the odd exponentiated skew-t (OE_{ST}) distribution over other two-parameter distributions are demonstrated using a real dataset. The odd exponentiated skew-t (OE_{ST}) distribution is compared with other competitive distributions such as the type-I half logistic skew-t (TIHLST), half logistic skew-t (HLST), Fréchet (FT), Pareto (PE), Lomax (LOMX), inverse Pareto (INVPE), type-I half logistic Burr X (TIHLBX) and skew Student-t (ST). The descriptive statistics of the dataset are provided in Table 3.

The dataset represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli. The dataset is given as follows:

0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55. This dataset has previously been used by Jamal *et al.* (2019), Umar *et al.* (2019), Leren and Abdullahi (2020) and Ampadu and Anafo (2019).

Table 3: Descriptive statistics of the dataset.

	n	**Mean**	**Median**	**SD**	**Skewness**	**Kurtosis**
Dataset	72	1.791	1.595	1.011	1.294	5.046

The performance measures are applied using the R-software package “AdequacyModel” to evaluate the fit of the distributions specified above. The distribution parameters are estimated using the maximum likelihood estimation procedure. The following performance measures: Hannan-Quinn information criterion (HQIC), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Anderson Darling (AD), Cramer-von Mises (CVM), Kolmogorov-Smirnov (K-S) statistic and its p-value are provided in Table 4. The distribution is of a good fit if all the performance measures are smaller and the p-value is larger. Lastly, Table 5 presents the 95% and 99% confidence intervals for the OE_{ST} distribution parameters.

Table 4: MLEs (SE) and performance measures.

Model	Par	MLEs (SEs)	AIC	BIC	HQIC	CVM	AD	K-S	p-value
OE_{ST}	$\hat{\alpha}$	0.0054(0.0022)	206.52	211.18	208.28	0.054	0.278	0.085	0.645
	$\hat{\lambda}$	0.0919(0.0397)							
TIHLST	$\hat{\alpha}$	0.6579(0.0889)	288.94	293.60	290.81	0.064	0.366	0.374	1.2e-09
	$\hat{\lambda}$	1.5812(0.5509)							
HLST	$\hat{\lambda}$	3.5701(0.7450)	297.13	299.46	298.06	0.099	0.579	0.456	3.8e-14
FT	$\hat{\alpha}$	1.1771(0.0819)	255.59	260.25	257.45	0.575	3.659	0.192	0.001
	$\hat{\lambda}$	1.0846(0.1125)							
PE	$\hat{\alpha}$	47.126(44.788)	245.71	250.38	247.58	0.078	0.524	0.296	3.3e-06
	$\hat{\theta}$	84.038(81.125)							
	$\hat{\beta}$	93.3635(73.8610)							
LOMX	$\hat{\alpha}$	40.0722(51.3010)	245.92	250.58	247.78	0.078	0.522	0.300	2.3e-06
	$\hat{\beta}$	0.0142(0.0183)							
INVPE	$\hat{\alpha}$	13.0025(11.3630)	258.46	263.12	260.32	0.320	2.160	0.257	8.6e-05
	$\hat{\beta}$	0.0980(0.0914)							
TIHLBX	$\hat{\alpha}$	-0.0788(0.0859)	209.059	216.05	211.85	0.164	1.030	0.104	0.377
	$\hat{\theta}$	0.7835(0.0762)							
	$\hat{\lambda}$	25.8169(45.9911)							
ST	$\hat{\lambda}$	5.7271(1.2579)	346.19	348.52	347.12	0.117	0.698	0.578	2.2e-16

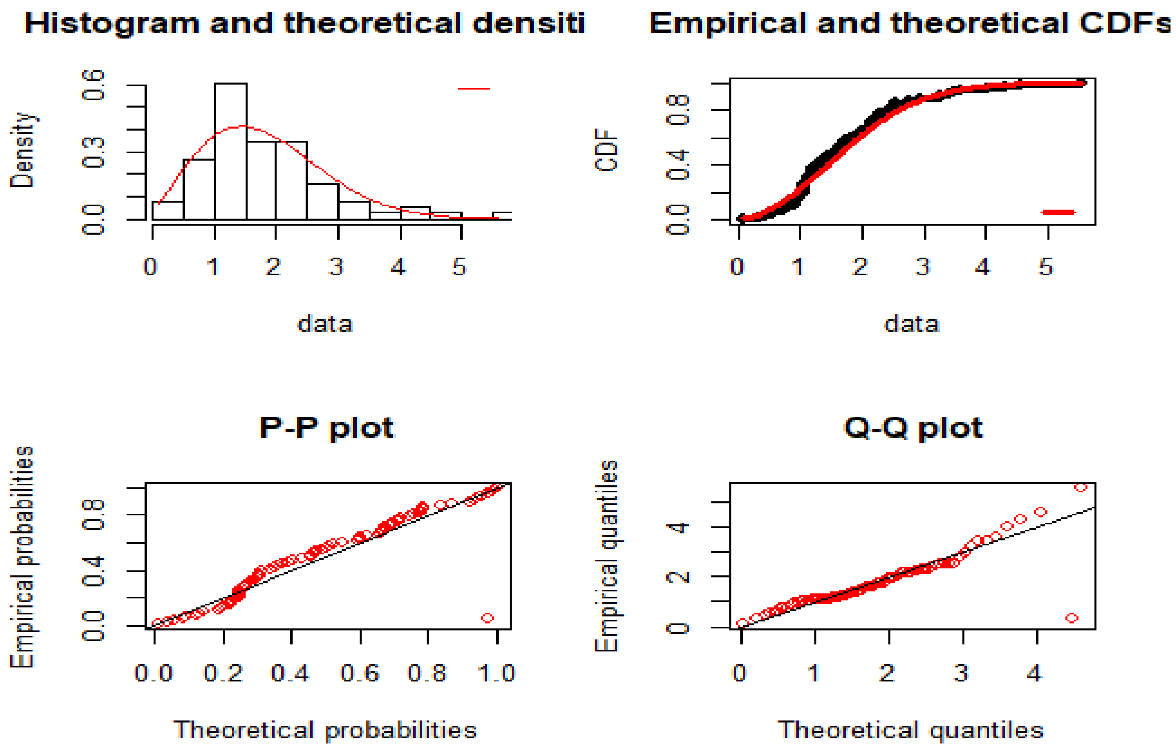

Figure 4: Fitted density function plot ***(top left panel)***, distribution function plot ***(top right panel)***, probability-probability (PP) plot ***(bottom left panel)*** and quantile-quantile (QQ) plot ***(bottom right panel)*** of the OE_{ST} distribution.

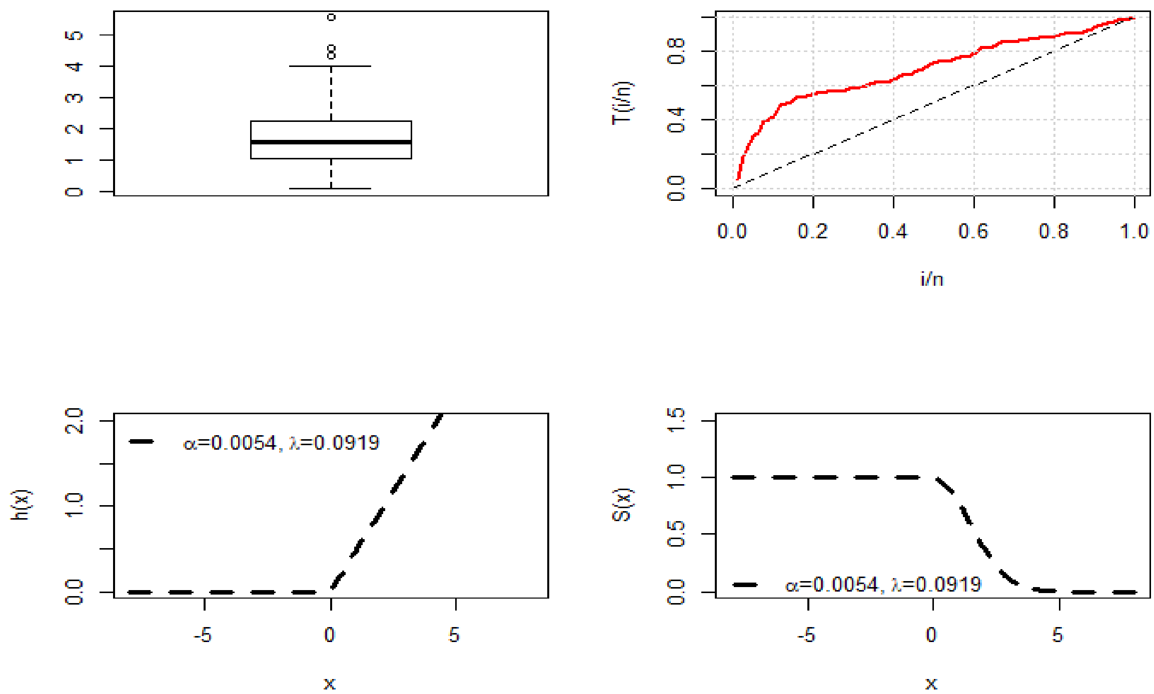

Figure 5: The Box plot ***(top left panel)***, total time of test (TTT) plot ***(top right panel)***, OE_{ST} hazard rate function plot ***(bottom left panel)*** and OE_{ST} survival function plot ***(bottom right panel).***

From the results in Table 4, the performance measures of the OE_{ST} distribution are smaller when compared to other fitted distributions, so we infer that the OE_{ST} distribution provides a good fit than the other distributions. The flexibility and fitness of the OE_{ST} distribution is visible from Figure 4. It is clear that OE_{ST} distribution provides an appropriate fit for the dataset based on the density function, distribution function, P-P plot and Q-Q plot in Figure 4. The TTT (total time on test) plot in Figure 5, shows that the dataset exhibits an increasing failure rate function and OE_{ST} is capable of accommodating increasing failure rates. Likewise, the Box plot of the dataset is shown in Figures 5.

Furthermore, the hazard rate and survival plots of the OE_{ST} distribution, using the parameter estimates in Table 4 are also depicted in Figure 5. The hazard rate shape based on the OE_{ST} parameter estimates is increasing and J-shaped. The J-shaped means that the OE_{ST} distribution tend to have some observations at one end, very few in the middle and a large number of observations at the other end which gives it the capability of handling skewed and heavy tail datasets. The results in Table 5, shows that the parameter estimates fall within the 95% and 99% confidence intervals.

Table 5. OE_{ST} distribution parameter estimates confidence intervals

CI	α	λ
95%	$[0.00109 \quad 0.00971]$	$[0.01409 \quad 0.16971]$
99%	$[-0.00025 \quad 0.01105]$	$[-0.01013 \quad 0.19393]$

6 CONCLUSION

This article presents a new two-parameter distribution known as the odd exponentiated skew-t (OE_{ST}) distribution using the odd exponentiated transformation. The flexibility of the skew-t distribution is improved using this transformation. This mixture representation is important to derive several structural properties of this distribution in full generality. Some of them are provided such as the ordinary and incomplete moments, quantile function, entropy, characteristic function and order statistics. The new distribution parameter estimates are derived using the maximum likelihood estimation (MLE) procedure and simulation study showed that the MLE performed well in estimating the parameters of the new distribution. The application using a real dataset indicates that the OE_{ST} distribution outperformed the other competing distributions. Future research study will compare the performance of the OE_{ST} error conditional distribution to existing error conditional distributions such as the normal distribution, Student-t distribution, generalized error distribution, and its skew variants in modeling and forecasting volatility using GARCH framework.

CONFLICTS OF INTEREST

Authors declare that there are no conflicts of interest.

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